Recovering the Electric Field from the Potential

For many systems of electric charges, it is easier to calculate the potential $V(x, y, z)$ than the electric field $\mathbf{E}(x, y, z)$. But once we know the potential, the electric field follows as (minus) its gradient,

$$\mathbf{E}(x, y, z) = -\nabla V(x, y, z),$$

or in components

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}.$$  

(1)

In these notes, I give a few examples of such calculations.

**Warm-up Example: The Coulomb Potential.**

As a warm-up example, let’s recover the Coulomb field from the Coulomb potential

$$V(r) = \frac{kQ}{r}.$$  

(3)

Let’s start with a general rule for the gradient of any spherically symmetric scalar field $s(r)$ which depends only on the radius

$$r = \sqrt{x^2 + y^2 + z^2}.$$  

(4)

By the *chain rule* for derivatives of functions of functions,

$$\frac{\partial s}{\partial x} = \frac{ds}{dr} \times \frac{\partial r}{\partial x}, \quad \frac{\partial s}{\partial y} = \frac{ds}{dr} \times \frac{\partial r}{\partial y}, \quad \frac{\partial s}{\partial z} = \frac{ds}{dr} \times \frac{\partial r}{\partial z},$$

(5)

or in vector notations

$$\nabla s(r) = \frac{ds}{dr} \times \nabla r.$$  

(6)

Now let’s calculate the gradient of the radius $r$ as a function of the Cartesian coordinates

$$r = \sqrt{x^2 + y^2 + z^2}.$$  

(4)
Using

\[ r^2 = x^2 + y^2 + z^2, \]  

we immediately obtain

\[ \frac{\partial r^2}{\partial x} = 2x, \quad \frac{\partial r^2}{\partial y} = 2y, \quad \frac{\partial r^2}{\partial z} = 2z, \]  

or in vector notations,

\[ \nabla r^2 = 2r. \]  

At the same time, by eq. (6)

\[ \nabla r^2 = \frac{dr^2}{dr} \times \nabla r = 2r \times \nabla r, \]  

hence

\[ \nabla r = \frac{1}{2r} \nabla r^2 = \frac{2r}{2r} = \frac{r}{r} = \hat{r}, \]  

the unit vector in the radial direction. Plugging this result into eq. (6), we arrive at

\[ \nabla s(r) = \frac{ds}{dr} \times \hat{r}. \]  

Applying this general formula to the Coulomb potential (3), we find

\[ \frac{d}{dr} \frac{kQ}{r} = -\frac{kQ}{r^2} \]  

and therefore

\[ \mathbf{E} = -\nabla V(r) = +\frac{kQ}{r^2} \times \hat{r}. \]
The Electric Dipole

Now consider a more interesting example — the electric dipole made out of two point charges, $+Q$ and $-Q$ at short distance $a$ from each other. The potential of this dipole is simply the algebraic sum of the Coulomb potentials due to each charges, thus

$$V(x, y, z) = \frac{kQ}{r_+(x, y, z)} - \frac{kQ}{r_-(x, y, z)}$$

(15)

where $r_+$ and $r_-$ are the distances from the two charges to the point $(x, y, z)$. Let me illustrate that with a diagram:

Let’s use coordinates with origin in the middle of the dipole and $z$ axis pointing in the direction of the dipole moment

$$\mathbf{d} = Qa.$$  

(17)

Then

$$r_+^2 = x^2 + y^2 + (z - \frac{a}{2})^2$$

$$= x^2 + y^2 + z^2 - az + \frac{a^2}{4}$$

$$= r^2 - ra \cos \theta + \frac{a^2}{4},$$

(18)

$$r_-^2 = x^2 + y^2 + (z + \frac{a}{2})^2$$

$$= x^2 + y^2 + z^2 + az + \frac{a^2}{4}$$

$$= r^2 + ra \cos \theta + \frac{a^2}{4},$$
hence

$$r_-^2 - r_+^2 = 2ar \cos \theta$$

(19)

and therefore

$$\frac{1}{r_+} - \frac{1}{r_-} = \frac{r_- - r_+}{r_-r_+} = \frac{r_-^2 - r_+^2}{r_-r_+(r_- + r_+)} = \frac{2ar \cos \theta}{r_-r_+(r_- + r_+)}.$$ \hspace{1cm} (20)

When we measure the potential at distance from the dipole which is much larger than the distance between its two charges, $r \gg a$, the denominator on the right hand side of eq. (20) may be approximated as $2r^3$, hence

$$\frac{1}{r_+} - \frac{1}{r_-} \approx \frac{2ar \cos \theta}{2r^3} = \frac{a \cos \theta}{r^2}$$ \hspace{1cm} (21)

and therefore

$$V = \frac{kQa \cos \theta}{r^2}.$$ \hspace{1cm} (22)

In terms of the dipole moment $d = Qa$,

$$V_{\text{dipole}} = \frac{kd \cos \theta}{r^2}.$$ \hspace{1cm} (23)

or in vector notations

$$V_{\text{dipole}} = \frac{k d \cdot \vec{r}}{r^2} = \frac{k d \cdot r}{r^3}.$$ \hspace{1cm} (24)

* * *

Now let’s find the electric field of the dipole by taking the gradient of the potential (24).
By Leibniz rule,
\[ \nabla \left( \frac{\mathbf{d} \cdot \mathbf{r}}{r^3} \right) = \frac{1}{r^3} \times \nabla (\mathbf{d} \cdot \mathbf{r}) + (\mathbf{d} \cdot \mathbf{r}) \times \nabla \frac{1}{r^3}, \]
(25)
where
\[ \nabla (\mathbf{d} \cdot \mathbf{r}) = \nabla (d_x x + d_y y + d_z z) = \text{vector}(d_x, d_y, d_z) = \mathbf{d} \]
(26)
while
\[ \nabla (r^{-3}) = \frac{dr^{-3}}{dr} \times \mathbf{\hat{r}} = -\frac{3}{r^4} \times \mathbf{\hat{r}} = -\frac{3r}{r^5}. \]
(27)
Consequently,
\[ \nabla \left( \frac{\mathbf{d} \cdot \mathbf{r}}{r^3} \right) = +\frac{\mathbf{d}}{r^3} - \frac{3(\mathbf{d} \cdot \mathbf{r}) \mathbf{r}}{r^5} = \frac{\mathbf{d} - 3(\mathbf{d} \cdot \mathbf{\hat{r}}) \mathbf{\hat{r}}}{r^3} \]
(28)
and therefore
\[ \mathbf{E}_{dipole} = -\nabla \left( \frac{k \mathbf{d} \cdot \mathbf{r}}{r^3} \right) = k \frac{3(\mathbf{d} \cdot \mathbf{\hat{r}}) \mathbf{\hat{r}} - \mathbf{d}}{r^3}. \]
(29)

In \((x, y, z)\) coordinates where the positive \(z\) axis is aligned with the dipole moment \(\mathbf{d}\), we have
\[ \mathbf{d} = (0, 0, d), \quad \mathbf{d} \cdot \mathbf{\hat{r}} = \frac{z}{r}, \quad \mathbf{\hat{r}} = \left( \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right), \]
(30)
hence
\[ \mathbf{E}_{x_dipole} = \frac{kd}{r^3} \times \frac{3xz}{r^2}, \]
\[ \mathbf{E}_{y_dipole} = \frac{kd}{r^3} \times \frac{3yz}{r^2}, \]
\[ \mathbf{E}_{z_dipole} = \frac{kd}{r^3} \times \frac{3z^2 - r^2}{r^2}. \]
(31)
Finally, the magnitude of the dipole field follows from
\[ (3xz)^2 + (3yz)^2 + (3z^2 - r^2)^2 = 9z^2(x^2 + y^2 + z^2) - 6z^2r^2 + r^4 = 3z^2r^2 + r^4, \]
(32)
hence
\[ |\mathbf{E}_{dipole}| = \frac{kd}{r^3} \times \sqrt{1 + 3\frac{z^2}{r^2}} = \frac{kd}{r^3} \times \sqrt{1 + 3\cos^2 \theta}. \]
(33)
The following two diagrams show both the electric field lines (in red) and the *equipotential surfaces* (in blue) of the electric dipole. The first diagram shows the immediate neighborhood of the dipole, \( r \sim a \):
The second diagram shows much larger distances $r \gg a$: 
THIN SPHERICAL SHELL OF CHARGES

For my third example, I am going to calculate the potential — and hence the electric field — or a uniformly charged thin spherical shell, without using the Gauss Law. Instead, I shall calculate the potential by directly integrating

\[ V(r) = \int \int \frac{k \, dQ(d'r')}{|r' - r|} \]  \hspace{1cm} (34)

over the charged sphere

The integral (34) is over the direction of the radius vector \( r' \) while its magnitude is fixed to the sphere’s radius \( R \); in spherical coordinates

\[ dQ = Q_{net} \times \frac{d\Omega}{4\pi} = Q_{net} \times d\theta \times \sin \theta \, d\phi. \] \hspace{1cm} (36)

If we point the \( \theta = 0 \) semi-axis along the direction of \( r \) — i.e., towards the point where we measure the potential — we may identify the latitude angle \( \theta \) with the angle between \( r \) and \( r' \) as on the figure (35). Consequently, by the cosine theorem

\[ |r' - r|^2 = R^2 + r^2 - 2Rr \cos \theta \] \hspace{1cm} (37)

and the integral (34) becomes

\[ V(r) = \frac{kQ_{net}}{4\pi} \times \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{\sin \theta}{\sqrt{R^2 + r^2 - 2Rr \cos \theta}} \] \hspace{1cm} (38)

This is a 2D integral, but fortunately nothing in the integrand depends on \( \phi \), so the inner
integral over $d\phi$ amounts to an overall factor of $2\pi$, thus

$$V(r) = \frac{kQ_{\text{net}}}{2} \times \int_0^\pi \frac{\sin \theta d\theta}{\sqrt{R^2 + r^2 - 2Rr \cos \theta}}. \tag{39}$$

To evaluate this integral, we change the integration variable from $\theta$ to the distance in the denominator

$$D = \sqrt{R^2 + r^2 - 2Rr \cos \theta}. \tag{40}$$

Taking squares of both sides of this expression and then taking differentials at constant $r$ and $R$, we obtain

$$2D \times dD = d(D^2) = d(R^2 + r^2 - 2Rr \cos \theta) = -2Rr \times d\cos \theta = +2Rr \times \sin \theta d\theta \tag{41}$$

and consequently

$$\frac{\sin \theta d\theta}{D} = \frac{dD}{rR}. \tag{42}$$

Plugging this formula into the integral (39), we obtain

$$V(r) = \frac{kQ_{\text{net}}}{2} \times \int_{D_1}^{D_2} \frac{dD}{rD} = \frac{kQ_{\text{net}}}{2} \times \frac{D_2 - D_1}{rR}. \tag{43}$$

In this formula, $D_1$ is the distance $D$ which obtains for $\theta = 0$ while $D_2$ is the distance for $\theta = \pi$. Specifically,

$$\cos(\theta = 0) = +1 \implies D_1 = \sqrt{R^2 + r^2 - 2Rr} = |R - r|, \tag{44}$$

$$\cos(\theta = \pi) = -1 \implies D_1 = \sqrt{R^2 + r^2 + 2Rr} = R + r,$$

and therefore

$$D_2 - D_1 = R + r - |R - r| = \begin{cases} 2r & \text{for } r < R, \\ 2R & \text{for } r > R. \end{cases} \tag{45}$$
Plugging this formula into eq. (43), we obtain

\[
\begin{align*}
\text{for } r < R, \quad V(r) &= \frac{kQ_{\text{net}}}{2} \times \frac{2r}{rR} = \frac{kQ_{\text{net}}}{R}, \\
\text{for } r > R, \quad V(r) &= \frac{kQ_{\text{net}}}{2} \times \frac{2R}{rR} = \frac{kQ_{\text{net}}}{r}.
\end{align*}
\] (46)

In other words, \textit{inside the sphere we have a constant potential}

\[
V(r) = \frac{kQ_{\text{net}}}{R} \tag{47}
\]

\textit{while outside the sphere we have the Coulomb potential}

\[
V(r) = \frac{kQ_{\text{net}}}{r}. \tag{48}
\]

Graphically,

\[
V \quad (49)
\]
Now that we know calculated the potential, the electric field follows by taking the gradient. Inside the sphere, $V = \text{const}$ implies $\mathbf{E} = -\nabla V = 0$, while outside the sphere we recover the Coulomb field

$$
\mathbf{E}(r) = -\nabla V(r) = -\frac{dV}{dr} \hat{r} = + \frac{kQ_{\text{net}}}{r^2} \hat{r}.
$$

(50)

Graphically,

\begin{align*}
\text{Graphically,}

\begin{tikzpicture}
\draw[->] (0,0) -- (5,0) node[below] {$r$};
\draw[->] (0,0) -- (0,5) node[above] {$\mathbf{E}$};
\draw[thick, red] (0,4) to[out=-90,in=180] (5,0);
\draw[thick, dotted] (0,4) to[out=90,in=180] (5,0);
\end{tikzpicture}
\end{align*}

(51)