

## Resistivity

Electric resistance  $R$  of a conductor depends on its size and shape as well as on the conducting material. The size- and shape-dependence was discovered by Georg Simon Ohm and is often treated as the second part of the Ohm's Law: **A conductor of length  $L$  and uniform cross-sectional area  $A$  — such as a wire — has resistance**

$$\frac{V}{A} = R = \frac{L}{A} \times \rho \quad (1)$$

where  $\rho$  is the **resistivity of the conducting material**. Note: the only geometric aspects of a wire which affect its resistance are the length  $L$  and the cross-sectional area  $A$ ; the shape of the cross-section — round, or square, or elliptic, or whatever — does not affect the resistance.

EXAMPLE: a copper wire of diameter  $d = \frac{1}{16}$  inch  $\approx 1.6$  mm and length  $L = 1.0$  mile  $\approx 1600$  m. The cross-sectional area of this wire is  $A = \pi(d/2)^2 \approx 2.0$  mm<sup>2</sup> while the resistivity of copper is  $\rho \approx 1.7 \cdot 10^{-8}$   $\Omega$  m, hence the resistance is

$$R = \frac{L}{A} \times \rho = \frac{1.6 \cdot 10^3 \text{ m}}{2.0 \cdot 10^{-6} \text{ m}^2} \times (1.7 \cdot 10^{-8} \text{ } \Omega \text{ m}) \approx 14 \text{ } \Omega. \quad (2)$$

Intuitively, we may understand the  $L/A$  dependence of resistance in terms of series or parallel circuits: Every meter of a wire is in series with every other meter, hence  $R \propto L$ . On the other hand, every mm<sup>2</sup> of the wire's cross-section is in parallel with every other mm<sup>2</sup>, hence  $R \propto 1/A$ . However, similar arguments do not work for hydraulics where viscous resistance to water flow through a pipe depends on shape of the pipe's cross-section and behaves like  $1/\text{radius}^4$  instead of  $1/\text{radius}^2$ . The reason for this difference is the friction between the flowing water and the pipe's walls, which makes the water near a wall flow slower than in the middle of the pipe. But in the electric case, the 'friction' is between the moving electrons and the non-moving ions throughout the conducting metal, so the 'walls' or rather outer boundaries of the wire do not matter. Consequently, the electric resistance is local in nature, which leads to its  $L/A$  dependence on the wire's length and cross-section.

To see how this works, let's rewrite eq. (1) in a local form which does not refer to the conductor's geometry at all. For this, we need to define the *electric current density*

$$J = \frac{I}{A} = \frac{\text{current}}{\text{cross-sectional area}}. \quad (3)$$

To be precise,  $\mathbf{J}$  is a vector pointing in the direction of the current; in a wire  $\mathbf{J}$  is parallel to the wire, but in a more complicated conductor  $\mathbf{J}$  can have different directions and different magnitudes in different parts of the conductor. Thus,  $\mathbf{J}(x, y, z)$  is a local vector quantity. For a non-uniform  $\mathbf{J}$ , the net current through some cross-section of the conductor is

$$I = \iint_{\text{crosssection}} \mathbf{J} \cdot d^2\mathbf{Area}. \quad (4)$$

Another important local quantity is the electric field  $\mathbf{E}(x, y, z)$ . In static electricity  $\mathbf{E}$  vanishes inside a conductor, but once the current flows  $\mathbf{E} \neq 0$ . In fact, *the local form of Ohm's Law relates the electric field to the current density,*

$$\mathbf{E} = \rho \mathbf{J}. \quad (5)$$

The  $\rho$  here is precisely the same resistivity as in eq. (1). Indeed, for a uniform wire of length  $L$  and cross-sectional area  $A$ , we have

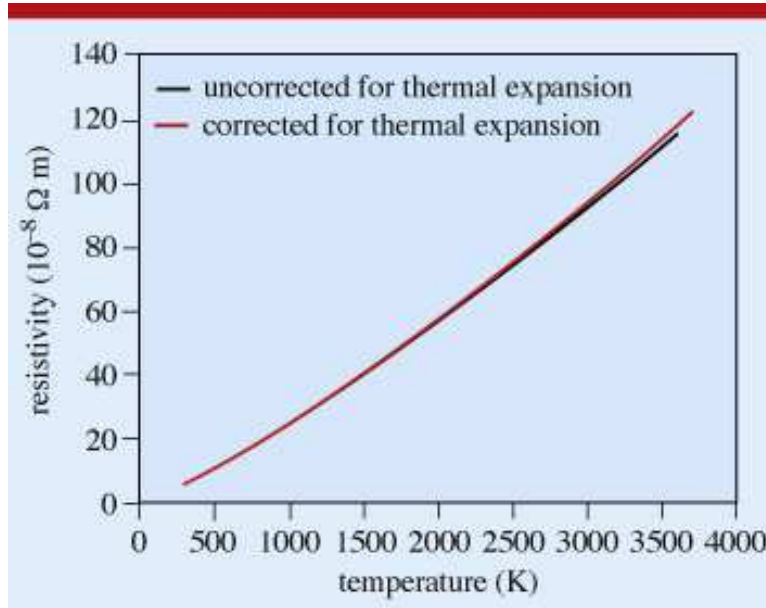
$$V = L \times E, \quad I = A \times J \quad (6)$$

and therefore

$$R = \frac{V}{I} = \frac{L \times E}{A \times J} = \frac{L}{A} \times \frac{E}{J} = \frac{L}{A} \times \rho. \quad (7)$$

Resistivities of some common metals, semiconductors, and insulators are given in tables 24.2–4 on textbook pages 866 and 871. There are also plenty of online resources, for example [http://www.engineeringtoolbox.com/resistivity-conductivity-d\\_418.html](http://www.engineeringtoolbox.com/resistivity-conductivity-d_418.html). Most metals have resistivities between  $1.6 \cdot 10^{-8} \Omega \text{ m}$  (silver) and  $10^{-6} \Omega \text{ m}$  (nichrome alloy) while most insulator have  $\rho$ 's ranging from  $2 \cdot 10^{+11} \Omega \text{ m}$  (polystyrene) to  $10^{+24} \Omega \text{ m}$  (teflon). The semiconductors have intermediate resistivities ranging from  $10^{-3} \Omega \text{ m}$  to  $10^{+3} \Omega \text{ m}$ , depending on purity and doping.

**Resistivities of metals increase with temperature.** For example, the figure below shows  $\rho(T)$  for the tungsten:



The light-emitting filaments in light bulbs are made of tungsten. They get very hot, about 2800 K or 4600 °F, which increases the resistivity tenfold compared to room temperature. Thus, once a light bulb has been lit for a couple of seconds, its electric resistance is ten times larger than the cold bulb's resistance!

For smaller changes of temperature, one often uses a linear approximation

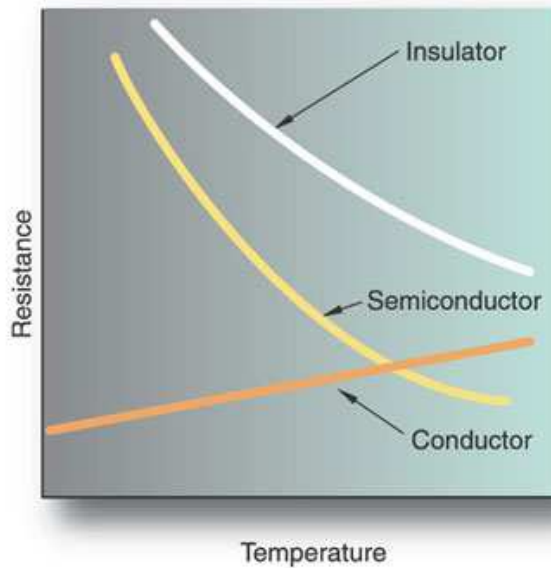
$$\rho(T) = \rho_0 \times (1 + \alpha \times (T - T_0)) \quad (8)$$

where  $T_0$  is some reference temperature (typically 20 °C),  $\rho_0 = \rho(T_0)$ , and  $\alpha$  is the *temperature coefficient of resistivity*. For example  $\alpha(\text{copper}) \approx 3.9 \cdot 10^{03} \text{ K}^{-1}$ , thus at 100 °C copper has resistivity

$$\rho(100^\circ\text{C}) = \rho(20^\circ\text{C}) \times (1 + \alpha \times (100^\circ - 20^\circ)) \approx \rho(20^\circ\text{C}) \times 1.31 \approx 2.2 \cdot 10^{-8} \text{ } \Omega \text{ m.} \quad (9)$$

On the other hand, at liquid nitrogen temperature 77 K (−196 °C or −321 °F), resistivity of copper drops to just 0.18 of its room temperature resistivity.

In contrast, **resistivities of semiconductors and insulators decrease with temperature:**



The inverse resistivity

$$\sigma = \frac{1}{\rho} \quad (10)$$

is called the *conductivity*. In terms of conductivity, the Ohm's Law becomes

$$\mathbf{J} = \sigma \mathbf{E} \quad (11)$$

## Drude Theory of Metals

Following the 1897 discovery of the electron by J. J. Thomson, Paul Drude and Hendrik Antoon Lorentz developed a theory of electric properties of metals based on *electron gas*. It's a classical theory — it does not include any quantum effects — and it ignores long-range interactions between electrons and ions, so it is not too accurate, but qualitatively it works surprisingly well. In these notes, I'll show how this theory — called the Drude model explains the electric resistivity of metals.

In the Drude model a metal comprises a lattice of positive ion cores and a gas of free electrons flying around between those cores; in a metal of valence  $z$ , each atom gives up  $z$  electrons to the gas. The electron gas behaves like an ideal gas: the electrons fly around at high speeds in random direction, and sometimes collide with the ion cores, as shown in [this you-tube video](#). The long-range interactions between the electrons, or between the electrons and the ions are neglected in the Drude model.

To understand the electric current and resistivity in Drude model, we need several collective properties of the electron gas. First, we need the *electron density*

$$n_e = \frac{\text{\#electrons}}{\text{volume}}. \quad (12)$$

For a metal of density  $\rho_m = \text{mass/volume}$ , atomic weight  $\mu$ , and valence  $z$ ,

$$n_e = z \times n_{\text{atoms}} = z \times \frac{\rho_m}{\mu/N_A} \quad (13)$$

where  $N_A = 6.02 \cdot 10^{23}$  atoms/mol =  $6.02 \cdot 10^{26}$  atoms/kmol. Please mind the units — mol v. kilomol, gram v. kilogram — when evaluating eq. (13). For example, copper has density  $\rho_m = 8960$  kg/m<sup>3</sup>, atomic weight  $\mu = 63.55$  kg/kmol, and valence  $z = 1$  (in metallic form), hence

$$n_e = (1 \text{ el/atom}) \times \frac{8960 \text{ kg/m}^3}{(63.55 \text{ kg/kmol})/(6.02 \cdot 10^{26} \text{ atom/kmol})} = 8.49 \cdot 10^{28} \text{ electrons/m}^3. \quad (14)$$

The next collective quantity we need is the *drift velocity of the electron gas*, which is the average velocity vector of an electron,

$$\mathbf{v}_d = \langle \mathbf{v} \rangle = \frac{1}{N} \sum_{i=1}^N \mathbf{v}_i. \quad (15)$$

Since the electrons fly in random directions, this drift velocity is much smaller than the

*average speed of an electron*

$$v_{\text{avg}} = \sqrt{\langle \mathbf{v}^2 \rangle} = \sqrt{\frac{1}{N} \sum \mathbf{v}_i^2}. \quad (16)$$

By analogy, the average velocity vector of an air molecule is the wind velocity,

$$\text{for the air, } \langle \mathbf{v}_{\text{molecule}} \rangle = \mathbf{v}_{\text{wind}}, \quad (17)$$

while the average speed of air molecule due to thermal motion is

$$\sqrt{\langle \mathbf{v}_{\text{molecule}}^2 \rangle} = \sqrt{\frac{3k_{\text{Boltzmann}}T}{m_{\text{molecule}}}} \sim 500 \text{ m/s}, \quad (18)$$

much faster than any wind! For the electron gas, the contrast is even larger: at room temperature, the average speed of an electron due to classical thermal motion is

$$v_{\text{avg}} = \sqrt{\frac{3kT}{m_e}} \approx 120,000 \text{ m/s}, \quad (19)$$

while the drift velocity is usually just a few millimeters per second!

The drift velocity  $\mathbf{v}_d$  and the electron density  $n_e$  determine the electric current density

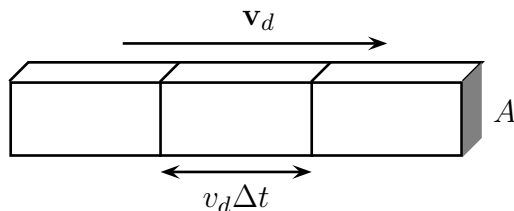
$$\mathbf{J} = -en_e\mathbf{v}_d \quad (20)$$

where  $-e$  is the electron's charge. For example, drift velocity  $v_d = 1.0 \text{ mm/s}$  in copper with  $n_e = 8.5 \cdot 10^{28} \text{ m}^{-3}$  gives rise to current density

$$J = (-1.6 \cdot 10^{-19} \text{ C}) \times (1.0 \cdot 10^{-3} \text{ m/s}) \times (8.5 \cdot 10^{28} \text{ m}^{-3}) \approx -14 \cdot 10^6 \text{ A/s}, \quad (21)$$

so a wire of cross-section  $A = 1 \text{ mm}^2$  carries current  $I = -14 \text{ A}$ . The  $-$  sign here indicates that the direction of the current is opposite to the direction of the electron gas's drift velocity.

To see where eq. (20) come from, consider the electron gas moving along a wire of cross-sectional area  $A$  at drift velocity  $v_d$ .



In time interval  $\Delta t$ , the gas travels through distance  $L = v_d \times \Delta t$ . Consequently, the volume of gas traversing any particular cross-section of the wire during this interval is

$$V = L \times A = v_d \times \Delta t \times A. \quad (22)$$

This volume has  $N = V \times n_e$  electrons of net charge

$$\Delta Q = -e \times N = -e \times n_e \times V = -en_e v_d \times A \times \Delta t. \quad (23)$$

Consequently, the electric current is

$$I = \frac{\Delta Q}{\Delta t} = -en_e v_d \times A \quad (24)$$

and the current density is

$$J = \frac{I}{A} = -en_e v_d. \quad (25)$$

One last property of the electron gas we need is the *mean time  $\tau$  an electron flies free between collisions with ions*. The Drude model assumes this time is independent from electron's velocity or position. It also assumes that each collision with an ion completely randomizes the direction of the electron's motion. Consequently, *right after a collision, the average velocity vector of an electron is zero*.

But what happens a short time  $\Delta t$  after a collision? In the absence of electric field, there are no forces acting on electrons between the collisions, so their velocities remain constant

until the next collision. But the electric field  $\mathbf{E}$  pushes each electron with a force  $\mathbf{F} = -e\mathbf{E}$ , which gives it acceleration

$$\mathbf{a} = -\frac{e}{m_e}\mathbf{E}. \quad (26)$$

Consequently, at time  $\Delta t$  after the last collision,

$$\mathbf{v}(i^{\text{th}} \text{ electron}) = \mathbf{v}_0(i^{\text{th}} \text{ electron}) + \mathbf{a}\Delta t, \quad (27)$$

so after averaging over the electrons

$$\langle \mathbf{v} \rangle = \langle \mathbf{v}_0 \rangle + \mathbf{a}\langle \Delta t \rangle = \mathbf{0} + \mathbf{a}\tau. \quad (28)$$

In other words, the electric field  $\mathbf{E}$  give the electron gas drift velocity

$$\mathbf{v}_d = \mathbf{a}\tau = -\frac{e\tau}{m_e}\mathbf{E}. \quad (29)$$

Another way to derive eq. (29) is to consider effect of collisions on the net momentum

$$\mathbf{P} = \sum_i m_e \mathbf{v}_i = Nm_e \mathbf{v}_d \quad (30)$$

of  $N$  electrons. (Note:  $N$  is very very large.) In very short time  $\Delta t \ll \tau$ ,

$$N_{\text{coll}} = N \times \frac{\Delta t}{\tau}$$

of the electrons will suffer a collision with an ion. On average, the momentum vector of an electron after a collision is zero, so the collisions reduce the net momentum  $\mathbf{P}$  by

$$-\Delta \mathbf{P}_{\text{coll}} = \frac{N_{\text{coll}}}{N} \mathbf{P} = \frac{\Delta t}{\tau} \mathbf{P}. \quad (31)$$

At the same time, the electrostatic forces  $\mathbf{F} = -e\mathbf{E}$  on each electron increase the net mo-



mentum by

$$\Delta \mathbf{P}_E = N \Delta t \mathbf{F} = -N \Delta t e \mathbf{E}. \quad (32)$$

Altogether, the two effects change the net momentum at the net rate

$$\frac{\Delta \mathbf{P}_{\text{net}}}{\Delta t} = -\frac{\mathbf{P}}{\tau} - Ne \mathbf{E}. \quad (33)$$

In a steady current, the net electron momentum should not change with time,  $\Delta \mathbf{P}_{\text{net}} = 0$ , which requires

$$-\frac{\mathbf{P}}{\tau} - Ne \mathbf{E} = \mathbf{0}. \quad (34)$$

In light of eq. (30), this means

$$-\frac{Nm_e \mathbf{v}_d}{\tau} - Ne \mathbf{E} = \mathbf{0} \quad (35)$$

and hence

$$\mathbf{v}_d = -\frac{e\tau}{m_e} \mathbf{E} \quad (36)$$

in accordance with eq. (29).

Finally, let us combine eq. (29) for the drift velocity of the electron gas and eq. (20) for the electric current density:

$$\mathbf{J} = -en_e \mathbf{v}_d = +\frac{e^2 n_e \tau}{m_e} \mathbf{E}. \quad (37)$$

Comparing this formula with the Ohm's Law

$$\mathbf{J} = \sigma \mathbf{E}, \quad (38)$$

we immediately obtain the electric conductivity

$$\sigma = \frac{e^2 n_e \tau}{m_e}, \quad (39)$$

or in terms of resistivity

$$\rho = \frac{m_e}{e^2 n_e \tau}. \quad (40)$$

Before we compare this prediction of the Drude model to the resistivities of real metals, we need an estimate of the mean time between collisions  $\tau$ . Drude and Lorentz assumes a fixed (for a particular metal) *mean free path*  $\lambda$  an electron travels between collisions, typically  $\lambda \sim$  a few interatomic distance, hence

$$\tau = \frac{\lambda}{v_{\text{avg}}} \quad (41)$$

where  $v_{\text{avg}}$  is the average speed of the electron's random motion. In classical statistical mechanics, this speed is given by eq. (19).

EXAMPLE: Let's assume a 30 Ångstrom mean free path,  $\lambda = 30 \cdot 10^{-10}$  m, and a thermal average speed for a room temperature, thus  $v_{\text{avg}} \approx 120,000$  m/s, and hence mean time between collisions  $\tau = 2.5 \cdot 10^{-14}$  s. Let's also assume a copper-like electron density  $n_e = 8.5 \cdot 10^{28}$  m<sup>-3</sup>. Plugging all these values into eq. (40), we obtain  $\rho = 1.7 \cdot 10^{-8}$  Ω m, very close to the real copper's resistivity.

Finally, consider the temperature dependence of the resistivity. In the Drude model the mean free path  $\lambda$  of an electron does not depend on the temperature, while the average speed  $v_{\text{avg}}$  increases like  $\sqrt{T}$ . Consequently, the mean time between collisions  $\tau$  decreases as

$$\tau \propto \frac{1}{\sqrt{T}}. \quad (42)$$

Nothing else in eq. (40) depends on the temperature, hence *in the Drude model, the resistivity increases with temperature as*

$$\rho(T) \propto \sqrt{T}. \quad (43)$$

Experimentally, the sign of this effect is correct, and the magnitude is in the right ballpark, but the more specific prediction  $\rho \propto \sqrt{T}$  is wrong.

The reason for this disagreement is the purely classical nature of the Drude–Lorentz theory. In the more accurate quantum theory, the average speed of electrons' random motion is larger than the classical thermal speed (19) but it does not depend on the temperature. On the other hand, the mean free path  $\lambda$  becomes shorter with increasing temperature due

to phonons which collide with the electrons. Consequently, the mean time between collisions decreases with temperature, but in a more complicated way than  $\tau \propto 1/\sqrt{T}$ , which leads for a more complicated formula for the  $\rho(T)$ .