1. The first problem is about Dirac's $\gamma$ matrices.

   (a) Verify $\left[S^{\kappa\lambda}, S^{\mu\nu}\right] = i\left(g^{\lambda\mu}S^{\kappa\nu} - g^{\lambda\nu}S^{\kappa\mu} - g^{\kappa\mu}S^{\lambda\nu} + g^{\kappa\nu}S^{\lambda\mu}\right)$.

   (b) Verify $M^{-1}(L)\gamma^\mu M(L) = L^\mu_{\ \nu}\gamma^\nu$ for $L = \exp(\theta)$ (*i.e.*, $L^\mu_{\ \nu} = \delta^\mu_\nu + \theta^\mu_{\ \nu} + \frac{1}{2}\theta^\mu_{\ \lambda}\theta^\lambda_{\ \nu} + \cdots$) and $M(L) = \exp\left(-\frac{i}{2}\theta_{\alpha\beta}S^{\alpha\beta}\right)$

   (c) Calculate $\{\gamma^\rho, \gamma^\lambda\gamma^\mu\gamma^\nu\}$, $[\gamma^\rho, \gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu]$ and $[S^{\rho\sigma}, \gamma^\lambda\gamma^\mu\gamma^\nu]$.

   (d) Show that $\gamma^\alpha\gamma_\alpha = 4$, $\gamma^\alpha\gamma^\nu\gamma_\alpha = -2\gamma^\nu$, $\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha = 4g^{\mu\nu}$ and $\gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\alpha = -2\gamma^\nu\gamma^\mu\gamma^\lambda$. Hint: use $\gamma^\alpha\gamma^\nu = 2g^{\nu\alpha} - \gamma^\nu\gamma^\alpha$ repeatedly.

   (e) Consider the electron's spinor field $\Psi(x)$ in an electomagnetic background. Show that the gauge-covariant Dirac equation $\left(i\gamma^\mu D_\mu + m\right)\Psi(x) = 0$ implies $\left(m^2 + D^2 + qF_{\mu\nu}S^{\mu\nu}\right)\Psi(x) = 0$.

2. The second problem is about the Lorentz group and its generators $\hat{J}^{\mu\nu}$. In 3–index notations, $\hat{J}^{ij} = \epsilon^{ij\ell}\hat{J}^\ell$ generate ordinary rotations while $\hat{J}^{0i} = -\hat{J}^{i0} = \hat{K}^i$ generate the Lorentz boosts. Let

$$\hat{\mathbf{J}}_\pm \;=\; \tfrac{1}{2}\left(\hat{\mathbf{J}} \;\pm\; i\hat{\mathbf{K}}\right). \tag{1}$$

   (a) Show that the $\hat{\mathbf{J}}_+$ and the $\hat{\mathbf{J}}_-$ commute with each other and that each satisfies the commutations relations of an angular momentum, $[\hat{J}^k_\pm, \hat{J}^\ell_\pm] = i\epsilon^{k\ell m}\hat{J}^m_\pm$.

   The "angular momentum" $\hat{\mathbf{J}}_+$ is non-hermitian and hence its finite irreducible representations are non-unitary analytic continuations of the spin–$j$ representations of a hermitian $\hat{\mathbf{J}}$. The same is true for the $\hat{\mathbf{J}}_- = \hat{\mathbf{J}}_+^\dagger$, so altogether, the finite irreducible representations of the Lorentz algebra are specified by two integer or half-integer 'spins' $j_+$ and $j_-$.

   The simplest non-trivial representations of the Lorentz algebra are the Weyl spinor ($j_+ = \frac{1}{2}, j_- = 0$) — a doublet where $\hat{\mathbf{J}}$ acts as $\frac{1}{2}\vec{\sigma}$ and $\hat{\mathbf{K}}$ as $-\frac{i}{2}\vec{\sigma}$ and the congugate Weyl 'anti-spinor' ($j_+ = 0, j_- = \frac{1}{2}$) where $\hat{\mathbf{J}}$ also acts as $\frac{1}{2}\vec{\sigma}$ but $\hat{\mathbf{K}}$ acts as $+\frac{i}{2}\vec{\sigma}$. Together the Weyl spinor and the Weyl antispinor comprise the Dirac spinor.

(b) Show that for any infinitesimal combination of a Lorentz boost $\vec{b}$ and rotation $\vec{\theta} \equiv \theta\mathbf{n}$,

$$\Psi'(x') \;=\; \Psi(x) \;+\; \begin{pmatrix} -\frac{i}{2}(\vec{\theta} - i\vec{b}) \cdot \vec{\sigma} & 0 \\ 0 & -\frac{i}{2}(\vec{\theta} + i\vec{b}) \cdot \vec{\sigma} \end{pmatrix} \Psi(x), \tag{2}$$

which means that a Dirac spinor indeed decomposes into a Weyl spinor and a Weyl antispinor.

Finite Lorentz transformations act on Weyl spinors as complex, unimodular (det = 1) but non-unitary two-by-two matrices. The group $SL(2,\mathbf{C})$ of such matrices is actually isomorphic to the Spin$(3,1)$ — the double cover of the continuous Lorentz group. (This is similar to Spin$(3) \cong SU(2)$.) Any $(j_+, j_-)$ representation of the Spin$(3,1)$ becomes in the $SL(2,\mathbf{C})$ terms a tensor $\Phi_{a_1 \ldots a_{(2j_+)}, \dot{a}_1 \ldots \dot{a}_{(2j_-)}}$, totally symmetric in its $2j_+$ un-dotted indices $a_1, \ldots, a_{(2j_+)}$ and separately totally symmetric in its $2j_-$ dotted indices $\dot{a}_1, \ldots, \dot{a}_{(2j_-)}$, transforming according to

$$\Phi'_{a_1 \ldots a_{(2j_+)}, \dot{a}_1 \ldots \dot{a}_{(2j_-)}} \;=\; U_{a_1}^{\;b_1} \cdots U_{a_{(2j_+)}}^{\;b_{(2j_+)}}\, U^{*\,\dot{b}_1}_{\dot{a}_1} \cdots U^{*\,\dot{b}_{(2j_-)}}_{\dot{a}_{(2j_-)}}\, \Phi_{b_1 \ldots b_{(2j_+)}, \dot{b}_1 \ldots \dot{b}_{(2j_-)}}\,. \tag{3}$$

The vector representation of the Lorentz group has $j_+ = j_- = \frac{1}{2}$. To cast the action of the Lorentz group in $SL(2,\mathbf{C})$ terms (3), consider $X^\mu \sigma_\mu = T - \mathbf{X} \cdot \vec{\sigma}$. (Here $\sigma^0 = 1$ while $\sigma^1$, $\sigma^2$ and $\sigma^3$ are the Pauli matrices.) Let

$$X'^\mu \sigma_\mu \;\equiv\; L^\mu_{\;\nu}(U)\, X^\nu \sigma_\mu \;=\; U\big(X^\mu \sigma_\mu\big) U^\dagger. \tag{4}$$

(c) Show that for any $SL(2,\mathbf{C})$ matrix $U$, eq. (4) indeed defines a Lorentz transform. (Hint: prove and use $\det(X^\mu \sigma_\mu) = X^2 \equiv X_\mu X^\mu$).
Also verify the group law, $L(U_2 U_1) = L(U_2) L(U_1)$.

(d) Verify explicitly that for $U = \exp\big(-\frac{i}{2}\theta\mathbf{n} \cdot \vec{\sigma}\big)$, $L(U)$ is a rotation by angle $\theta$ around axis $\mathbf{n}$ while for $U = \exp\big(-\frac{1}{2} r\mathbf{n} \cdot \vec{\sigma}\big)$, $L(U)$ is a boost of rapidity $r$ ($\beta = \tanh r$, $\gamma = \cosh r$) in the direction $\mathbf{n}$.

3. Finally, consider the relation between Lorentz transformations of the fields and of the particles. In mechanics (classical or quantum), one must distinguish between two opposite kinds of rotations, namely coordinate-frame rotations of bodies and body-frame rotations of coordinate systems. For the Lorentz transformations of fields and particles, there is a similar distinction between the particle-frame and field-frame Lorentz transforms.

   For example, consider a real (hermitian) scalar quantum field

$$\hat{\Phi}(x) \;=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \left[ e^{-ipx}\,\hat{a}(p) \;+\; e^{+ipx}\,\hat{a}^\dagger(p) \right]_{p^0 \equiv E_\mathbf{p}} \tag{5}$$

   (where $\hat{a}(p)$ stands for the $\hat{a}_\mathbf{p}(t=0)$ and ditto for the $\hat{a}^\dagger(p)$). A field-frame Lorentz transform $L$ acts on this field according to

$$\hat{\Phi}'(x') \;\equiv\; \hat{\mathcal{D}}^\dagger(L)\,\hat{\Phi}(x')\,\hat{\mathcal{D}}(L) \;=\; \hat{\Phi}(x = L^{-1}x') \tag{6}$$

   while the corresponding particle-frame transform acts precisely in reverse:

$$\hat{\mathcal{D}}(L)\,\hat{\Phi}(x)\,\hat{\mathcal{D}}^\dagger(L) \;=\; \hat{\Phi}(Lx). \tag{7}$$

   In both cases $\hat{\mathcal{D}}(L) = \exp\bigl(\frac{i}{2}\theta_{\alpha\beta}\hat{J}^{\alpha\beta}\bigr)$ is a unitary operator representing the lorentz transform $L$ in the Fock space of the quantum field theory.

   (a) Show that (7) implies

$$\begin{aligned} \hat{\mathcal{D}}(L)\bigl(\sqrt{2p^0}\,\hat{a}(p)\bigr)\hat{\mathcal{D}}^\dagger(L) \;&=\; \sqrt{2(Lp)^0}\,\hat{a}(Lp), \\ \hat{\mathcal{D}}(L)\bigl(\sqrt{2p^0}\,\hat{a}^\dagger(p)\bigr)\hat{\mathcal{D}}^\dagger(L) \;&=\; \sqrt{2(Lp)^0}\,\hat{a}^\dagger(Lp), \\ &\text{and hence} \\ \hat{\mathcal{D}}(L)\,|p\rangle \;&=\; |Lp\rangle\,, \\ \hat{\mathcal{D}}(L)\,|p_1,p_2\rangle \;&=\; |Lp_1,Lp_2\rangle\,, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{8}$$

   (Thus *particle*-frame Lorentz transform.)

Now consider a generic quantum field

$$\hat{\phi}_a(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \Big[ e^{-ipx} f_a(p,s)\, \hat{a}(p,s) + e^{+ipx} h_a(p,s)\, \hat{b}^\dagger(p,s) \Big]_{p^0 \equiv E_{\mathbf{p}}} \tag{9}$$

where $e^{-ipx} f_a(p,s)$ and $e^{+ipx} h_a(p,s)$ are independent plane-wave solutions of the free field equation for the $\phi_a$, whatever that might be. We assume complex (*i.e.*, non-hermitian) $\hat{\phi}_a(x)$; otherwise we would have $\hat{b}^\dagger(p,s) = \hat{a}^\dagger(p,s)$ and $h_a(p,s) = f_a^*(p,s)$.

The field $\hat{\phi}_a(x)$ transforms according to some representation $M_a{}^b(L)$ of the Lorentz symmetry, thus

$$\hat{\phi}'_a(x') \equiv \hat{\mathcal{D}}^\dagger(L)\, \hat{\phi}_a(x')\, \hat{\mathcal{D}}(L) = \sum_b M_a{}^b(L)\, \hat{\phi}_b(x = L^{-1}x') \tag{10}$$

in the field frame and

$$\hat{\mathcal{D}}(L)\, \hat{\phi}_a(x)\, \hat{\mathcal{D}}^\dagger(L) = \sum_b M_a{}^b(L^{-1})\, \hat{\phi}_b(Lx) \tag{11}$$

in the particle frame.

(b) Verify that formula (11) is consistent with the group Law for the Lorentz symmetry, $\hat{\mathcal{D}}(L_2 L_1) = \hat{\mathcal{D}}(L_2)\, \hat{\mathcal{D}}(L_1)$.

(c) A particle-frame Lorentz transform should act on particle — or antiparticle — quantum numbers according to

$$\hat{\mathcal{D}}(L)\, |p, \pm, s\rangle = \sum_{s'} C_{s,s'}(L,p)\, \big|Lp, \pm, s'\big\rangle\,. \tag{12}$$

Show that eqs. (11) and (12) are consistent with each other if and only if

$$\begin{aligned} f_a(Lp, s') &= \sum_b \sum_s M_a{}^b(L)\, C^*_{s,s'}(L,p)\, f_b(p,s), \\ h_a(Lp, s') &= \sum_b \sum_s M_a{}^b(L)\, C_{s,s'}(L,p)\, h_b(p,s). \end{aligned} \tag{13}$$