

1. First, a little exercise about Weyl spinors. As discussed in class, a Dirac spinor field $\Psi(x)$ is physically equivalent to two left-handed Weyl spinor fields $\chi(x)$ and $\varphi(x)$. In Weyl basis,

$$\Psi(x) \equiv \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} = \begin{pmatrix} \chi(x) \\ -\sigma^2 \varphi^*(x) \end{pmatrix}. \quad (1)$$

- (a) Show that in terms of the Weyl spinor fields, the Dirac Lagrangian becomes

$$\mathcal{L} \equiv \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi = i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + i\varphi^\dagger \bar{\sigma}^\mu \partial_\mu \varphi + m(\varphi^\top \sigma^2 \chi + \varphi^\dagger \sigma^2 \chi^*) \quad (2)$$

(up to a total derivative). Note that $\chi(x)$, $\varphi(x)$, $\chi^*(x)$ and $\varphi^*(x)$ are *fermionic* fields, so in the classical limit they *anticommute* with each other rather than commute. Thus, $\varphi^\top \sigma^2 \chi = +\chi^\top \sigma^2 \varphi$ even though the σ^2 matrix is antisymmetric.

- (b) Verify that each term in the Lagrangian (2) is separately invariant (up to a total derivative) under the combined Parity + Charge Conjugation symmetry $\widehat{\mathcal{C}}\widehat{\mathcal{P}}$.

2. The rest of this homework concerns Dirac spinor fields and their discrete symmetries. We begin with the charge-conjugation properties of Dirac bilinears $\widehat{\Psi}\Gamma\widehat{\Psi}$.

- (a) Show that $\widehat{\mathcal{C}}\widehat{\Psi}\Gamma\widehat{\Psi}\widehat{\mathcal{C}} = \widehat{\Psi}\Gamma^c\widehat{\Psi}$ where $\Gamma^c = \gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2$.

Hint: Mind the anticommutativity of the fermionic fields.

- (b) Calculate Γ^c for all 16 independent matrices Γ and find out which Dirac bilinears are \mathcal{C} -even and which are \mathcal{C} -odd.

- (c) Consider a bound state of a fermion and an antifermion, *e.g.* a positronium state or a neutral meson. As argued in class, the parity of such bound state is $P = -(-1)^L$. Show that the C-parity of this state is $C = (-1)^S(-1)^L$ and use this fact to explain why the decay mode and the lifetime of a 1S positronium state depend on its spin.

Hint: $\hat{a}^\dagger \hat{b}^\dagger = -\hat{b}^\dagger \hat{a}^\dagger$.

3. The time-reversal symmetry involves an *anti-linear*, *anti-unitary* operator \hat{T} that invert directions of all particle momenta and spins,[★]

$$\hat{T} |\text{particle type, } \mathbf{p}, s\rangle = (\text{phase}) |\text{same particle type, } -\mathbf{p}, -s\rangle. \quad (3)$$

The phase factor here combines an arbitrary but fixed overall phase η with a phase factor inherent in spin reversal: $\widehat{\mathcal{SR}} |m_s\rangle = i^{2m_s} |-m_s\rangle$ in the \hat{S}_z eigenbasis, or more generally,

$$\widehat{\mathcal{SR}} |\xi\rangle = i^{2S} e^{-\pi i \hat{S}_y} |\xi\rangle^*. \quad (4)$$

In particular, for spin $\frac{1}{2}$ particles, $\widehat{\mathcal{SR}} |\xi\rangle = \sigma_2 |\xi\rangle^*$, in agreement with the rule $\xi_{-s} = \sigma_2 \xi_s^*$ we used in class. Note however that reversing the spin twice results in a rotation by 2π , $\widehat{\mathcal{SR}}^2 = e^{-2\pi i \hat{S}_y} = (-1)^{2S}$, which is trivial for integral spins but changes the overall sign of the spin state for half-integral spins. Consequently, the time-reversal operator in the Fock space satisfies

$$\hat{T}^2 = (-1)^F \equiv \text{rotation by } 2\pi. \quad (5)$$

In this exercise, we consider time-reversal of the Dirac spinor field. First, we need a lemma:

$$i^{2m_s} u^*(-\mathbf{p}, -m_s) = -i\gamma^1\gamma^3 u(+\mathbf{p}, +m_s), \quad i^{2m_s} v^*(-\mathbf{p}, -m_s) = -i\gamma^1\gamma^3 v(+\mathbf{p}, +m_s). \quad (6)$$

(a) Prove this lemma.

In terms of electronic and positronic creation and annihilation operators, eq. (3) means

$$\begin{aligned} \hat{T} \hat{a}^\dagger(\mathbf{p}, m_s) \hat{T}^{-1} &= (\mp i) i^{2m_s} \hat{a}^\dagger(-\mathbf{p}, -m_s), \\ \hat{T} \hat{b}^\dagger(\mathbf{p}, m_s) \hat{T}^{-1} &= (\pm i) i^{2m_s} \hat{b}^\dagger(-\mathbf{p}, -m_s), \\ \hat{T} \hat{a}(\mathbf{p}, m_s) \hat{T}^{-1} &= (\pm i) i^{2m_s} \hat{a}(-\mathbf{p}, -m_s), \\ \hat{T} \hat{b}(\mathbf{p}, m_s) \hat{T}^{-1} &= (\mp i) i^{2m_s} \hat{b}(-\mathbf{p}, -m_s), \end{aligned} \quad (7)$$

where the spin-independent phase factors $\pm i$ make for a consistent time-reversal of the Dirac spinor field.

★ Please see J. J. Sakurai *Modern Quantum Mechanics*, §4.4 for a discussion of time reversal in general and spin reversal in particular.

(b) Use lemma (6) and eqs. (7) to show that

$$\hat{\mathcal{T}}\hat{\Psi}(\mathbf{x}, t)\hat{\mathcal{T}}^{-1} = \pm\gamma^1\gamma^3\Psi(\mathbf{x}, -t). \quad (8)$$

(c) Next, consider the Dirac bilinears $\hat{\bar{\Psi}}\Gamma\hat{\Psi}$ and show that $\hat{\mathcal{T}}\hat{\bar{\Psi}}\Gamma\hat{\Psi}\hat{\mathcal{T}}^{-1} = \hat{\bar{\Psi}}\Gamma^t\hat{\Psi}$ where $\Gamma^t = \gamma^3\gamma^1\Gamma^*\gamma^1\gamma^3$.

(d) Calculate Γ^t for all 16 independent matrices Γ and find out which Dirac bilinears are \mathcal{T} -even and which are \mathcal{T} -odd.

(e) Verify the \mathcal{T} -invariance of the Dirac action.

4. Finally, consider the combined $\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}$ symmetry of the Dirac field and verify that for any bilinear operator $\hat{\mathcal{O}}(x) = \hat{\bar{\Psi}}(x)\Gamma\hat{\Psi}(x)$,

$$\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}\hat{\mathcal{O}}(x)[\hat{\mathcal{C}}\hat{\mathcal{P}}\hat{\mathcal{T}}]^{-1} = \hat{\mathcal{O}}^\dagger(-x) \times (-1)^{\#\text{Lorentz indices in } \hat{\mathcal{O}}}. \quad (9)$$

Actually, eq. (9) holds for any physically measurable operator $\hat{\mathcal{O}}(x)$ in any legitimate quantum field theory — this is the famous CPT theorem — but the exercise is limited to the Dirac bilinear operators only.