

Problem 1(a):

As explained in class, the properly normalized wave function of the N particle state $|(\alpha_1, \alpha_2, \dots, \alpha_N)\rangle = |\{n_\beta\}_\beta\rangle$ is

$$\begin{aligned}
 \Psi(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) &= \frac{1}{\sqrt{\# \text{ distinct permutations}}} \sum_{\substack{\text{distinct} \\ \text{permutations} \\ \alpha'_1, \dots, \alpha'_N \\ \text{of } \alpha_1, \dots, \alpha_N}} \varphi_{\alpha'_1}(\mathbf{x}_1) \cdots \varphi_{\alpha'_N}(\mathbf{x}_N) \\
 &= \frac{1}{\sqrt{\# \text{ distinct permutations}}} \times \frac{1}{(\# \text{ redundant permutations})} \sum_{\substack{\text{all} \\ \text{permutations} \\ \alpha'_1, \dots, \alpha'_N \\ \text{of } \alpha_1, \dots, \alpha_N}} \varphi_{\alpha'_1}(\mathbf{x}_1) \cdots \varphi_{\alpha'_N}(\mathbf{x}_N).
 \end{aligned} \tag{S.1}$$

Combinatorically, the number of redundant *i.e.*, trivial permutation of the set of $\alpha_1, \dots, \alpha_N$ is $\prod_\beta n_\beta!$ while the number of distinct permutations is therefore $N! / \prod_\beta n_\beta!$. Furthermore, any permutation of the functions $\varphi_{\alpha_1}, \dots, \varphi_{\alpha_N}$ in a product $\varphi_{\alpha_1}(\mathbf{x}_1) \cdots \varphi_{\alpha_N}(\mathbf{x}_N)$ is equivalent to an opposite permutation of the coordinates $\mathbf{x}_1, \dots, \mathbf{x}_N$. Substituting these facts into the second line of eq. (S.1) immediately yields eq. (2) of the problem set. *Q.E.D.*

Problem 1(b):

First, let us assume $|N, \Psi\rangle = |(\alpha_1, \alpha_2, \dots, \alpha_N)\rangle$ and hence $\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N)$ of the form (2). Consequently,

$$|N+1, \Psi'\rangle \stackrel{\text{def}}{=} \hat{a}_\alpha^\dagger |N, \Psi\rangle = \sqrt{n_\alpha + 1} |(\alpha_1, \alpha_2, \dots, \alpha_N, \alpha)\rangle, \tag{S.2}$$

so Ψ' is also of the form (2). Specifically,

$$\begin{aligned}
 \Psi'(\mathbf{x}_1, \dots, \mathbf{x}_{N+1}) &\propto \sum_{\nu_1, \dots, \nu_{N+1}=1, \dots, (N+1)}^{\text{distinct}} \varphi_{\alpha_1}(\mathbf{x}_{\nu_1}) \cdots \varphi_{\alpha_N}(\mathbf{x}_{\nu_N}) \varphi_\alpha(\mathbf{x}_{\nu_{N+1}}) \\
 &\langle\langle \text{renaming } \nu_{N+1} \rightarrow i \text{ and regrouping terms} \rangle\rangle \\
 &= \sum_{i=1}^{N+1} \varphi_\alpha(\mathbf{x}_i) \sum_{\nu_1, \dots, \nu_N=1, \dots, \cancel{i}, \dots, (N+1)}^{\text{distinct}} \varphi_{\alpha_1}(\mathbf{x}_{\nu_1}) \cdots \varphi_{\alpha_N}(\mathbf{x}_{\nu_N}) \\
 &\propto \sum_{i=1}^{N+1} \varphi_\alpha(\mathbf{x}_i) \Psi(\mathbf{x}_1, \dots, \cancel{\mathbf{x}}_i, \dots, \mathbf{x}_{N+1}),
 \end{aligned} \tag{S.3}$$

exactly as in eq. (3) we are trying to prove.

As to the overall numerical factor in eq. (3), we need the ratio between the coefficients in eqs. (2) for the Ψ and the Ψ' :

$$\frac{\text{coeff.}(\Psi')}{\text{coeff.}(\Psi)} = \frac{\left(N'! \times \prod_{\beta} n'_{\beta}!\right)^{-1/2}}{\left(N! \times \prod_{\beta} n_{\beta}!\right)^{-1/2}} = \sqrt{\frac{N!}{N'!}} \times \prod_{\beta} \sqrt{\frac{n_{\beta}!}{n'_{\beta}!}} = \frac{1}{\sqrt{N+1}} \times \frac{1}{\sqrt{n_{\alpha}+1}}. \quad (\text{S.4})$$

Combining this ratio with the $\sqrt{n_{\alpha}+1}$ factor in eq. (S.2) we arrive at the $1/\sqrt{N+1}$ factor in eq. (3).

Note that the last coefficient depends only on the total particle number N but not on any coincidences between $\alpha_1, \dots, \alpha_N$. Consequently, eq. (3) applies not only to wave functions of the form (2) but also to all linear combinations of such wave functions (note linearity of the creation operator $\hat{a}_{\alpha}^{\dagger}$). And since the states $|(\alpha_1, \alpha_2, \dots, \alpha_N)\rangle$ comprise a complete basis of the N -boson Hilbert space \mathcal{H}_N^B , eq. (3) applies to all (properly symmetric) N -particle wave functions Ψ . *Q.E.D.*

Problem 1(c):

The action of an annihilation operator \hat{a}_{α} follows from that of its hermitian conjugate — the creation operator $\hat{a}_{\alpha}^{\dagger}$. Indeed, for any N -particle state $|N, \Psi\rangle$ and any $N-1$ particle state $|N-1, \tilde{\Psi}\rangle$, one has

$$\langle N-1, \tilde{\Psi} | \hat{a}_{\alpha} | N, \Psi \rangle = \langle N, \Psi | \hat{a}_{\alpha}^{\dagger} | N-1, \tilde{\Psi} \rangle^*, \quad (\text{S.5})$$

which in terms of the wave functions becomes

$$\begin{aligned} & \int d\mathbf{x}_1 \cdots \int d\mathbf{x}_{N-1} \tilde{\Psi}^*(\mathbf{x}_1, \dots, \mathbf{x}_{N-1}) \Psi''(\mathbf{x}_1, \dots, \mathbf{x}_{N-1}) \\ &= \left[\int d\mathbf{x}_1 \cdots \int d\mathbf{x}_N \Psi^*(\mathbf{x}_1, \dots, \mathbf{x}_N) \tilde{\Psi}'(\mathbf{x}_1, \dots, \mathbf{x}_N) \right]^* \\ &= \int d\mathbf{x}_1 \cdots \int d\mathbf{x}_N \Psi(\mathbf{x}_1, \dots, \mathbf{x}_N) \times \frac{1}{\sqrt{N}} \sum_{i=1}^N \varphi_{\alpha}^*(\mathbf{x}_i) \tilde{\Psi}^*(\mathbf{x}_1, \dots, \cancel{\mathbf{x}}_i, \dots, \mathbf{x}_N) \\ & \quad \langle\langle \text{by symmetry of the } \Psi \rangle\rangle \end{aligned}$$

$$\begin{aligned}
&= \int d\mathbf{x}_1 \cdots \int d\mathbf{x}_N \Psi(\mathbf{x}_1, \dots, \mathbf{x}_N) \times \sqrt{N} \varphi_\alpha^*(\mathbf{x}_N) \tilde{\Psi}^*(\mathbf{x}_1, \dots, \mathbf{x}_{N-1}) \\
&= \int d\mathbf{x}_1 \cdots \int d\mathbf{x}_{N-1} \tilde{\Psi}^*(\mathbf{x}_1, \dots, \mathbf{x}_{N-1}) \times \sqrt{N} \int d\mathbf{x}_N \varphi_\alpha^*(\mathbf{x}_N) \Psi(\mathbf{x}_1, \dots, \mathbf{x}_{N-1}, \mathbf{x}_N).
\end{aligned}$$

This relation must hold true for all $\tilde{\Psi}$, which immediately implies eq. (4). *Q.E.D.*

Problem 2(a):

In ordinary one-particle QM, matrix elements of an operator define its wave-function action according to

$$[\hat{A}_1 \Psi](\mathbf{x}) = \sum_{\alpha, \beta} \langle \alpha | \hat{A}_1 | \beta \rangle \varphi_\alpha(\mathbf{x}) \int d\mathbf{x}' \varphi_\beta^*(\mathbf{x}') \Psi(\mathbf{x}'). \quad (\text{S.6})$$

Formula (7) is a straightforward generalization of this action to a one-particle operator acting only upon the i^{th} particle of an N particle system. Consequently, for any N particle wave function,

$$[\hat{A}_{\text{tot}} \Psi](\mathbf{x}_1, \dots, \mathbf{x}_N) = \sum_{\alpha, \beta} \langle \alpha | \hat{A}_1 | \beta \rangle \times \sum_{i=1}^N \varphi_\alpha(\mathbf{x}_i) \int d\mathbf{x}'_i \varphi_\beta^*(\mathbf{x}'_i) \Psi(\mathbf{x}_1, \dots, \mathbf{x}'_i, \dots, \mathbf{x}_N). \quad (\text{S.7})$$

Now consider the second quantized formalism. Thanks to the symmetry of any legitimate N -boson wavefunction, we may re-write eq. (4) for the action of an annihilation operator \hat{a}_β as

$$[\hat{a}_\beta \Psi](\mathbf{x}_1, \dots, \cancel{\mathbf{x}}_i, \dots, \mathbf{x}_N) = \sqrt{N} \int d\mathbf{x}'_i \varphi_\beta^*(\mathbf{x}'_i) \Psi(\mathbf{x}_1, \dots, \mathbf{x}'_i, \dots, \mathbf{x}_N). \quad (\text{S.8})$$

Next, we act on this expression with a creation operator \hat{a}_α^\dagger , which according to eq. (3) yields

$$[\hat{a}_\alpha^\dagger \hat{a}_\beta \Psi](\mathbf{x}_1, \dots, \mathbf{x}_N) = \sum_{i=1}^N \varphi_\alpha(\mathbf{x}_i) \int d\mathbf{x}'_i \varphi_\beta^*(\mathbf{x}'_i) \Psi(\mathbf{x}_1, \dots, \mathbf{x}'_i, \dots, \mathbf{x}_N). \quad (\text{S.9})$$

Substituting this formula into eq. (6) immediately leads for an expression for the wave function of the state $\hat{A} |N, \Psi\rangle$ that is identical with eq. (S.7). *Q.E.D.*

Note that the above derivation applies to the states with $N > 0$ particles only. For the vacuum state $|0\rangle$, we use $\hat{a}_\beta |0\rangle = 0$, which immediately implies $\hat{A} |0\rangle = 0$, in agreement with $\hat{A}_{\text{tot}}(0 \text{ particles}) \equiv 0$.

Problem 2(b):

For a two-particle operator, the analogue of eq. (S.6) is given by

$$[\hat{A}_2 \Psi](\mathbf{x}_1, \mathbf{x}_2) = \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha \otimes \beta | \hat{A}_2 | \gamma \otimes \delta \rangle \varphi_\alpha(\mathbf{x}_1) \varphi_\beta(\mathbf{x}_2) \int d\mathbf{x}'_1 \int d\mathbf{x}'_2 \varphi_\gamma^*(\mathbf{x}'_1) \varphi_\delta^*(\mathbf{x}'_2) \Psi(\mathbf{x}'_1, \mathbf{x}'_2). \quad (\text{S.10})$$

At this point, Bose statistics is optional, hence the use of un-symmetrized matrix elements $\langle \alpha \otimes \beta | \hat{A}_2 | \gamma \otimes \delta \rangle$. However, similarity between the two particles leads to

$$\langle \alpha \otimes \beta | \hat{A}_2 | \gamma \otimes \delta \rangle = \langle \beta \otimes \alpha | \hat{A}_2 | \delta \otimes \gamma \rangle, \quad (\text{S.11})$$

which in turn implies that if the wave function $\Psi(\mathbf{x}_1, \mathbf{x}_2)$ happens to be symmetric with respect to $\mathbf{x}_1 \leftrightarrow \mathbf{x}_2$, then the $[\hat{A}_2 \Psi](\mathbf{x}_1, \mathbf{x}_2)$ is also symmetric.

For an additive two-particle operator (8) acting on an $N \geq 2$ particle system, we proceed exactly as in the one-particle case and obtain

$$\begin{aligned} [\hat{A}_{\text{tot}} \Psi](\mathbf{x}_1, \dots, \mathbf{x}_N) &= \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha \otimes \beta | \hat{A}_2 | \gamma \otimes \delta \rangle \\ &\times \sum_{i \neq j} \varphi_\alpha(\mathbf{x}_i) \varphi_\beta(\mathbf{x}_j) \int d\mathbf{x}'_i \int d\mathbf{x}'_j \varphi_\gamma^*(\mathbf{x}'_i) \varphi_\delta^*(\mathbf{x}'_j) \Psi(\mathbf{x}_1, \dots, \mathbf{x}'_i, \dots, \mathbf{x}'_j, \dots, \mathbf{x}_N), \end{aligned} \quad (\text{S.12})$$

where the indices i and j run from 1 to N and by a slight abuse of notations we write $\Psi(\mathbf{x}_1, \dots, \mathbf{x}'_i, \dots, \mathbf{x}'_j, \dots, \mathbf{x}_N)$ regardless of whether $i < j$ or $i > j$.

In the second quantized formalism, we again assume a totally symmetric $\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N)$, which allows us to use eq. (S.8) twice and write for any $i \neq j$ (in the 1 to N range)

$$\begin{aligned} [\hat{a}_\gamma \hat{a}_\delta \Psi](\mathbf{x}_1, \dots, \cancel{\mathbf{x}}_i, \dots, \cancel{\mathbf{x}}_j, \dots, \mathbf{x}_N) \\ = \sqrt{N(N-1)} \int d\mathbf{x}'_i \int d\mathbf{x}'_j \varphi_\gamma^*(\mathbf{x}'_i) \varphi_\delta^*(\mathbf{x}'_j) \Psi(\mathbf{x}_1, \dots, \mathbf{x}'_i, \dots, \mathbf{x}'_j, \dots, \mathbf{x}_N). \end{aligned} \quad (\text{S.13})$$

Consequently,

$$\begin{aligned}
& [\hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta \Psi](\mathbf{x}_1, \dots, \mathbf{x}_N) \\
&= \sum_{i=1}^N \varphi_\alpha(\mathbf{x}_i) \sum_{j \neq i} \varphi_\beta(\mathbf{x}_j) \times \int d\mathbf{x}'_i \int d\mathbf{x}'_j \varphi_\gamma^*(\mathbf{x}'_i) \varphi_\delta^*(\mathbf{x}'_j) \Psi(\mathbf{x}_1, \dots, \mathbf{x}'_i, \dots, \mathbf{x}'_j, \dots, \mathbf{x}_N);
\end{aligned} \tag{S.14}$$

comparing this formula with eq. (S.12) immediatly proves eq. (9).

To complete the proof, consider the special cases of the vacuum and of the one particle states. In both cases, there are no two-particle operators and hence $\hat{A}_{\text{tot}} \equiv 0$. In the second quantized formalism, $\hat{A} |N < 2, \text{whatever}\rangle = 0$ follows from all $N < 2$ states being annihilated by all products of two annihilation operators: Indeed, $\hat{a}_\delta |0\rangle = 0$ while $\hat{a}_\delta |N = 1, \Psi\rangle \propto |0\rangle$ and hence $\hat{a}_\gamma \hat{a}_\delta |N = 1, \Psi\rangle = 0$. *Q.E.D.*

Problem 3(a):

This is a simple exercise of the Leibnitz rule for commutators:

$$\begin{aligned}
[\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger] &= [\hat{a}_\alpha^\dagger, \hat{a}_\gamma^\dagger] \hat{a}_\beta + \hat{a}_\alpha^\dagger [\hat{a}_\beta, \hat{a}_\gamma^\dagger] = 0 + \hat{a}_\alpha^\dagger \delta_{\beta,\gamma} = \delta_{\beta,\gamma} \hat{a}_\alpha^\dagger, \\
[\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\delta] &= [\hat{a}_\alpha^\dagger, \hat{a}_\delta] \hat{a}_\beta + \hat{a}_\alpha^\dagger [\hat{a}_\beta, \hat{a}_\delta] = -\delta_{\alpha,\delta} \hat{a}_\beta + 0 = -\delta_{\alpha,\delta} \hat{a}_\beta, \\
[\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger \hat{a}_\delta] &= [\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger] \hat{a}_\delta + \hat{a}_\gamma^\dagger [\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\delta] = \delta_{\beta,\gamma} \hat{a}_\alpha^\dagger \hat{a}_\delta - \delta_{\alpha,\delta} \hat{a}_\gamma^\dagger \hat{a}_\beta.
\end{aligned} \tag{S.15}$$

Problem 3(b):

Given

$$\hat{A} = \sum_{\alpha,\beta} \langle \alpha | \hat{A}_1 | \beta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta \quad \text{and} \quad \hat{B} = \sum_{\gamma,\delta} \langle \gamma | \hat{B}_1 | \delta \rangle \hat{a}_\gamma^\dagger \hat{a}_\delta,$$

we immediately have

$$\begin{aligned}
[\hat{A}, \hat{B}] &= \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha | \hat{A}_1 | \beta \rangle \langle \gamma | \hat{B}_1 | \delta \rangle [\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger \hat{a}_\delta] \\
&\llcorner\text{using (S.15)}\llcorner \\
&= \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha | \hat{A}_1 | \beta \rangle \langle \gamma | \hat{B}_1 | \delta \rangle (\delta_{\beta, \gamma} \hat{a}_\alpha^\dagger \hat{a}_\delta - \delta_{\alpha, \delta} \hat{a}_\gamma^\dagger \hat{a}_\beta) \\
&= \sum_{\alpha, \delta} \hat{a}_\alpha^\dagger \hat{a}_\delta \times \sum_{\beta=\gamma} \langle \alpha | \hat{A}_1 | \gamma \rangle \langle \gamma | \hat{B}_1 | \delta \rangle - \sum_{\beta, \gamma} \hat{a}_\gamma^\dagger \hat{a}_\beta \times \sum_{\alpha=\delta} \langle \gamma | \hat{B}_1 | \alpha \rangle \langle \alpha | \hat{A}_1 | \beta \rangle \\
&= \sum_{\alpha, \delta} \hat{a}_\alpha^\dagger \hat{a}_\delta \langle \alpha | \hat{A}_1 \hat{B}_1 | \delta \rangle - \sum_{\beta, \gamma} \hat{a}_\gamma^\dagger \hat{a}_\beta \langle \gamma | \hat{B}_1 \hat{A}_1 | \beta \rangle \\
&\llcorner\text{renaming summation indices}\llcorner \\
&= \sum_{\alpha, \beta} \hat{a}_\alpha^\dagger \hat{a}_\beta \times (\langle \alpha | \hat{A}_1 \hat{B}_1 | \beta \rangle - \langle \alpha | \hat{B}_1 \hat{A}_1 | \beta \rangle) \\
&= \sum_{\alpha, \beta} \hat{a}_\alpha^\dagger \hat{a}_\beta \times \langle \alpha | ([\hat{A}_1, \hat{B}_1] = \hat{C}_1) | \beta \rangle \equiv \hat{C}.
\end{aligned} \tag{S.16}$$

Problem 3(c):

Again we use the Leibnitz rule:

$$\begin{aligned}
[\hat{a}_\mu^\dagger \hat{a}_\nu, \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta] &= [\hat{a}_\mu^\dagger \hat{a}_\nu, \hat{a}_\alpha^\dagger] \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta + \hat{a}_\alpha^\dagger [\hat{a}_\mu^\dagger \hat{a}_\nu, \hat{a}_\beta^\dagger] \hat{a}_\gamma \hat{a}_\delta + \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger [\hat{a}_\mu^\dagger \hat{a}_\nu, \hat{a}_\gamma] \hat{a}_\delta + \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma [\hat{a}_\mu^\dagger \hat{a}_\nu, \hat{a}_\delta] \\
&= \delta_{\nu\alpha} \hat{a}_\mu^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta + \delta_{\nu\beta} \hat{a}_\alpha^\dagger \hat{a}_\mu^\dagger \hat{a}_\gamma \hat{a}_\delta - \delta_{\mu\gamma} \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\nu \hat{a}_\delta - \delta_{\mu\delta} \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\nu.
\end{aligned} \tag{S.17}$$

Problem 3(d):

Proceeding as in 3(b), we calculate

$$\begin{aligned}
[\hat{A}, \hat{B}] &= \sum_{\mu, \nu, \alpha, \beta, \gamma, \delta} \langle \mu | \hat{A}_1 | \nu \rangle \langle \alpha \otimes \beta | \hat{B}_2 | \gamma \otimes \delta \rangle [\hat{a}_\mu^\dagger \hat{a}_\nu, \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta] \\
&\langle\langle \text{using (S.17)} \rangle\rangle \\
&= \sum_{\mu, \beta, \gamma, \delta} \hat{a}_\mu^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta \times \sum_{\nu} \langle \mu | \hat{A}_1 | \nu \rangle \langle \nu \otimes \beta | \hat{B}_2 | \gamma \otimes \delta \rangle \\
&\quad + \sum_{\alpha, \mu, \gamma, \delta} \hat{a}_\alpha^\dagger \hat{a}_\mu^\dagger \hat{a}_\gamma \hat{a}_\delta \times \sum_{\nu} \langle \mu | \hat{A}_1 | \nu \rangle \langle \alpha \otimes \nu | \hat{B}_2 | \gamma \otimes \delta \rangle \\
&\quad - \sum_{\alpha, \beta, \nu, \delta} \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\nu \hat{a}_\delta \times \sum_{\mu} \langle \alpha \otimes \beta | \hat{B}_2 | \mu \otimes \delta \rangle \langle \mu | \hat{A}_1 | \nu \rangle \\
&\quad - \sum_{\alpha, \beta, \gamma, \nu} \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\nu \times \sum_{\mu} \langle \alpha \otimes \beta | \hat{B}_2 | \gamma \otimes \mu \rangle \langle \mu | \hat{A}_1 | \nu \rangle \\
&\langle\langle \text{renaming summation indices} \rangle\rangle \\
&= \sum_{\alpha, \beta, \gamma, \delta} \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta \times C_{\alpha, \beta, \gamma, \delta},
\end{aligned} \tag{S.18}$$

where

$$\begin{aligned}
C_{\alpha, \beta, \gamma, \delta} &= \sum_{\lambda} \langle \alpha | \hat{A}_1 | \lambda \rangle \langle \lambda \otimes \beta | \hat{B}_2 | \gamma \otimes \delta \rangle + \sum_{\lambda} \langle \beta | \hat{A}_1 | \lambda \rangle \langle \alpha \otimes \lambda | \hat{B}_2 | \gamma \otimes \delta \rangle \\
&\quad - \sum_{\lambda} \langle \alpha \otimes \beta | \hat{B}_2 | \lambda \otimes \delta \rangle \langle \lambda | \hat{A}_1 | \gamma \rangle - \sum_{\lambda} \langle \alpha \otimes \beta | \hat{B}_2 | \gamma \otimes \lambda \rangle \langle \lambda | \hat{A}_1 | \delta \rangle \\
&= \langle \alpha \otimes \beta | \left(\hat{A}_1(1^{\text{st}}) \hat{B}_2 + \hat{A}_1(2^{\text{nd}}) \hat{B}_2 - \hat{B}_2 \hat{A}_1(1^{\text{st}}) - \hat{B}_2 \hat{A}_1(2^{\text{nd}}) \right) | \gamma \otimes \delta \rangle \\
&= \langle \alpha \otimes \beta | \left[\left(\hat{A}_1(2^{\text{nd}}) + \hat{A}_1(2^{\text{nd}}) \right), \hat{B}_2 \right] | \gamma \otimes \delta \rangle \equiv \langle \alpha \otimes \beta | \hat{C}_2 | \gamma \otimes \delta \rangle.
\end{aligned} \tag{S.19}$$

Consequently, $[\hat{A}, \hat{B}] = \hat{C}$. *Q.E.D.*