

Problem 1(a):

As discussed in class for the massless case (EM),  $\partial\mathcal{L}/\partial(\partial_\mu A_\nu) = -F^{\mu\nu}$ .

Clearly,  $\partial\mathcal{L}/\partial(A_\nu) = +m^2 A^\nu - J^\nu$ . Hence, the Euler–Lagrange field equation is

$$-\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} + \frac{\partial\mathcal{L}}{\partial(A_\nu)} \equiv \partial_\mu F^{\mu\nu} + m^2 A^\nu - J^\nu = 0, \quad (\text{S.1})$$

or in terms of  $A^\nu$  and their explicit derivatives,

$$\partial^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) + m^2 A^\nu - J^\nu = 0. \quad (\text{S.2})$$

Problem 1(b):

Take the divergence  $\partial_\nu$  of the field equation (S.2); the first two terms cancel out while the rest becomes

$$m^2 \partial_\nu A^\nu - \partial_\nu J^\nu = 0. \quad (\text{S.3})$$

In the massless case, this equation enforces the current conservation  $\partial_\nu J^\nu = 0$  regardless of the 4–vector potential  $A^\nu(x)$ , but there is no such constraint in the massive case at hand. Instead, eq. (S.3) simply relates the current divergence to the 4–potential divergence. In particular, *if the current happens to satisfy  $\partial_\nu J^\nu$ , then — and only then — eq. (S.3) requires  $\partial_\nu A^\nu = 0$  as well.* Consequently, the field equation (S.2) simplifies to  $(\partial^2 + m^2)A^\nu = J^\nu$ .  
*Q.E.D.*

Problem 1(c):

As in (a),  $\partial\mathcal{L}/\partial(\partial^\mu A^\nu) = -F_{\mu\nu}$ .

In particular,  $\partial\mathcal{L}/\partial(\partial^0 A^i) = -F_{0i} = -E^i$  while  $\partial\mathcal{L}/\partial(\partial^0 A^0) = -F_{00} = 0$ .

Problem 1(d):

In terms of the Hamiltonian and Lagrangian densities, eq. (3) means

$$\mathcal{H} = -\dot{\mathbf{A}} \cdot \mathbf{E} - \mathcal{L}. \quad (3')$$

In terms of  $\mathbf{A}$ ,  $\mathbf{E}$  and  $A_0$ ,

$$\begin{aligned} \dot{\mathbf{A}} &= -\mathbf{E} - \nabla A_0, \\ -\dot{\mathbf{A}} \cdot \mathbf{E} &= \mathbf{E}^2 + \mathbf{E} \cdot \nabla A_0, \\ \mathcal{L} &= \frac{1}{2} (\mathbf{E}^2 - (\nabla \times \mathbf{A})^2) + \frac{1}{2} m^2 (A_0^2 - \mathbf{A}^2) - (A_0 J_0 - \mathbf{A} \cdot \mathbf{J}). \end{aligned} \quad (S.4)$$

Consequently,

$$\mathcal{H} = \frac{1}{2} \mathbf{E}^2 + \mathbf{E} \cdot \nabla A_0 - \frac{1}{2} m^2 A_0^2 + A_0 J_0 + \frac{1}{2} (\nabla \times \mathbf{A})^2 + \frac{1}{2} m^2 \mathbf{A}^2 - \mathbf{A} \cdot \mathbf{J}, \quad (S.5)$$

which immediately leads to eq. (4) (after one integration by parts). *Q.E.D.*

Problem 1(e):

Expanding the left hand side of eq. (5) gives us

$$\frac{\delta H}{\delta A_0(\mathbf{x})} \equiv \frac{\partial \mathcal{H}}{\partial (A_0)} - \nabla_i \frac{\partial \mathcal{H}}{\partial (\nabla_i A_0)} = -m^2 A_0 + J_0 - \nabla_i E^i,$$

hence

$$A_0(\mathbf{x}, t) = \frac{J_0 - \nabla \cdot \mathbf{E}}{m^2} \quad (S.6)$$

everywhere in spacetime. Please note that there are no derivatives at the left hand side of this formula.

Likewise, expanding the right hand side of the first eq. (6) yields

$$\frac{\delta H}{\delta E^i(\mathbf{x})} \equiv \frac{\partial \mathcal{H}}{\partial(E^i)} - \nabla_j \frac{\partial \mathcal{H}}{\partial(\nabla_j E^i)} = E^i + \nabla_i A_0$$

and hence

$$\frac{\partial}{\partial t} \mathbf{A}(\mathbf{x}, t) = -\mathbf{E} - \nabla A_0. \quad (\text{S.7})$$

Similarly,

$$\frac{\delta H}{\delta A^i(\mathbf{x})} \equiv \frac{\partial \mathcal{H}}{\partial(A^i)} - \nabla_j \frac{\partial \mathcal{H}}{\partial(\nabla_j A^i)} = m^2 A^i - J^i - \nabla_j (\epsilon^{jik} (\nabla \times \mathbf{A})^k)$$

and hence

$$\frac{\partial}{\partial t} \mathbf{E}(\mathbf{x}, t) = m^2 \mathbf{A} - \mathbf{J} + \nabla \times (\nabla \times \mathbf{A}). \quad (\text{S.8})$$

Problem 1(f):

In the 3-space notations, the Euler–Lagrange field equations (S.1) become

$$\nabla \cdot \mathbf{E} - m^2 A_0 = J_0, \quad (\text{S.9})$$

$$-\dot{\mathbf{E}} + \nabla \times \mathbf{B} + m^2 \mathbf{A} = \mathbf{J}, \quad (\text{S.10})$$

where

$$\mathbf{E} \stackrel{\text{def}}{=} -\dot{\mathbf{A}} - \nabla A_0, \quad (\text{S.11})$$

$$\mathbf{B} \stackrel{\text{def}}{=} \nabla \times \mathbf{A}. \quad (\text{S.12})$$

Clearly, eq. (S.9) is equivalent to eq. (S.6) while eq. (S.10) is equivalent to eq. (S.8) (provided  $\mathbf{B}$  is defined as in eq. (S.12)). Finally, eq. (S.11) is equivalent to eq. (S.7), although their origins differ: In the Lagrangian formalism, eq. (S.11) is the definition of the  $\mathbf{E}$  field in terms of  $A_0$ ,  $\mathbf{A}$  and their derivatives, while in the Hamiltonian formalism,  $\mathbf{E}$  is an independent conjugate field and eq. (S.7) is the dynamical equation of motion for the  $\dot{\mathbf{A}}$ . *Q.E.D.*

Problem 2:

Assume generic commutation relations

$$\begin{aligned}
[\hat{E}^j(\mathbf{x}, t), \hat{E}^k(\mathbf{x}', t' = t)] &= \hat{f}^{jk}(\mathbf{x} - \mathbf{x}'), \\
[\hat{B}^j(\mathbf{x}, t), \hat{B}^k(\mathbf{x}', t' = t)] &= \hat{g}^{jk}(\mathbf{x} - \mathbf{x}'), \\
[\hat{E}^j(\mathbf{x}, t), \hat{B}^k(\mathbf{x}', t' = t)] &= \hat{h}^{jk}(\mathbf{x} - \mathbf{x}').
\end{aligned} \tag{S.13}$$

Commuting the quantum electromagnetic fields with the Hamiltonian (9), we obtain

$$\begin{aligned}
i \frac{\partial}{\partial t} \hat{E}^j(\mathbf{x}, t) &= [\hat{E}^j(\mathbf{x}, t), \hat{H}(t)] \\
&= \int d^3 \mathbf{x}' \left( \frac{1}{2} \left\{ \hat{f}^{jk}(\mathbf{x} - \mathbf{x}'), \hat{E}^k(\mathbf{x}', t) \right\} + \frac{1}{2} \left\{ \hat{h}^{jk}(\mathbf{x} - \mathbf{x}'), \hat{B}^k(\mathbf{x}', t) \right\} \right) \\
i \frac{\partial}{\partial t} \hat{B}^j(\mathbf{x}, t) &= [\hat{B}^j(\mathbf{x}, t), \hat{H}(t)] \\
&= \int d^3 \mathbf{x}' \left( \frac{1}{2} \left\{ \hat{g}^{jk}(\mathbf{x} - \mathbf{x}'), \hat{B}^k(\mathbf{x}', t) \right\} - \frac{1}{2} \left\{ \hat{h}^{kj}(\mathbf{x}' - \mathbf{x}), \hat{E}^k(\mathbf{x}', t) \right\} \right)
\end{aligned} \tag{S.14}$$

Comparing these Heisenberg equations with the time-dependent Maxwell equations (8), we see that we want

$$\begin{aligned}
\frac{1}{2} \left\{ \hat{f}^{jk}(\mathbf{x} - \mathbf{x}'), \hat{E}^k(\mathbf{x}', t) \right\} &= 0, \\
\frac{1}{2} \left\{ \hat{h}^{jk}(\mathbf{x} - \mathbf{x}'), \hat{B}^k(\mathbf{x}', t) \right\} &= i \epsilon^{j\ell k} \frac{\partial}{\partial x^\ell} \delta^{(3)}(\mathbf{x} - \mathbf{x}') \hat{B}^k(\mathbf{x}', t), \\
\frac{1}{2} \left\{ \hat{g}^{jk}(\mathbf{x} - \mathbf{x}'), \hat{B}^k(\mathbf{x}', t) \right\} &= 0, \\
\frac{1}{2} \left\{ \hat{h}^{kj}(\mathbf{x}' - \mathbf{x}), \hat{E}^k(\mathbf{x}', t) \right\} &= i \epsilon^{j\ell k} \frac{\partial}{\partial x^\ell} \delta^{(3)}(\mathbf{x} - \mathbf{x}') \hat{E}^k(\mathbf{x}', t),
\end{aligned} \tag{S.15}$$

and the simplest way to satisfy these requirements is to have  $c$ -number valued (rather than operator-valued) functions

$$\begin{aligned}
f^{jk}(\mathbf{x} - \mathbf{x}') &\equiv g^{jk}(\mathbf{x} - \mathbf{x}') \equiv 0, \\
h^{jk}(\mathbf{x} - \mathbf{x}') &= h^{kj}(\mathbf{x}' - \mathbf{x}) = i \epsilon^{j\ell k} \frac{\partial}{\partial x^\ell} \delta^{(3)}(\mathbf{x} - \mathbf{x}').
\end{aligned} \tag{S.16}$$

In other words

$$\begin{aligned}
[\hat{E}^j(\mathbf{x}, t), \hat{E}^k(\mathbf{x}', t' = t)] &= 0, \\
[\hat{B}^j(\mathbf{x}, t), \hat{B}^k(\mathbf{x}', t' = t)] &= 0, \\
[\hat{E}^j(\mathbf{x}, t), \hat{B}^k(\mathbf{x}', t' = t)] &= i\epsilon^{j\ell k} \frac{\partial}{\partial x^\ell} \delta^{(3)}(\mathbf{x} - \mathbf{x}') = i\epsilon^{k\ell j} \frac{\partial}{\partial x'^\ell} \delta^{(3)}(\mathbf{x}' - \mathbf{x}).
\end{aligned} \tag{S.17}$$

It remains to verify the consistency of these commutation relation with the transversality of the free electromagnetic fields:  $\nabla \cdot \hat{\mathbf{E}}$  and  $\nabla \cdot \hat{\mathbf{B}}$  vanish (*cf.* eq. (7)) and thus have to commute with everything. On the other hand, the commutation relations of the  $\nabla \cdot \hat{\mathbf{E}}$  and  $\nabla \cdot \hat{\mathbf{B}}$  follow from those of the individual components of the fields:

$$[\nabla \cdot \hat{\mathbf{E}}(\mathbf{x}), \text{whatever}(\mathbf{x}')] = \frac{\partial}{\partial x^j} [\hat{E}^j(\mathbf{x}), \text{same whatever}(\mathbf{x}')] \tag{S.18}$$

and ditto for the magnetic field. Applying this rule to the commutation relations (S.17) (and making use of  $\epsilon^{j\ell k} \nabla_j \nabla_\ell = 0$ ), we have

$$\begin{aligned}
[\nabla \cdot \hat{\mathbf{E}}(\mathbf{x}), \hat{E}^k(\mathbf{x}')] &= 0, \\
[\nabla \cdot \hat{\mathbf{E}}(\mathbf{x}), \hat{B}^k(\mathbf{x}')] &= i\epsilon^{j\ell k} \nabla_j \nabla_\ell \delta^{(3)}(\mathbf{x} - \mathbf{x}') = 0, \\
[\nabla \cdot \hat{\mathbf{B}}(\mathbf{x}), \hat{B}^k(\mathbf{x}')] &= 0, \\
[\nabla \cdot \hat{\mathbf{B}}(\mathbf{x}), \hat{E}^k(\mathbf{x}')] &= -i\epsilon^{j\ell k} \nabla_j \nabla_\ell \delta^{(3)}(\mathbf{x} - \mathbf{x}') = 0.
\end{aligned} \tag{S.19}$$

*Q.E.D.*