

Problem 1(a):

Use product-of-exponentials formula

$$\forall \hat{A}, \hat{B} : e^{\hat{A}} e^{\hat{B}} = \exp \left( \hat{A} + \hat{B} + \frac{1}{2} [\hat{A}, \hat{B}] + \frac{1}{12} [(\hat{A} - \hat{B}), [\hat{A}, \hat{B}]] + \dots \right). \quad (\text{S.1})$$

In particular,

$$e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} = \exp \left( \alpha \hat{a}^\dagger - \alpha^* \hat{a} + \frac{1}{2} \alpha \alpha^* + 0 \right),$$

hence

$$|\alpha\rangle \stackrel{\text{def}}{=} e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} |0\rangle = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} |0\rangle = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} |0\rangle. \quad (\text{S.2})$$

(Note  $e^{-\alpha^* \hat{a}} |0\rangle = |0\rangle$  since  $\hat{a} |0\rangle = 0$ .)

Next,  $[\hat{a}, \hat{a}^\dagger] = 1$  implies that for any function  $f(\hat{a}^\dagger)$ ,  $[\hat{a}, f(\hat{a}^\dagger)] = f'(\hat{a}^\dagger)$ . In particular,  $[\hat{a}, e^{\alpha \hat{a}^\dagger}] = \alpha e^{\alpha \hat{a}^\dagger}$  or in other words,  $(\hat{a} - \alpha) e^{\alpha \hat{a}^\dagger} = e^{\alpha \hat{a}^\dagger} \hat{a}$  and hence  $(\hat{a} - \alpha) |\alpha\rangle \propto e^{\alpha \hat{a}^\dagger} \hat{a} |0\rangle = 0$ .  
*Q.E.D.*

Problem 1(b):

For any *normal-ordered* product of creation and annihilation operators — *i.e.*, a product in which all creation operators are to the right of all annihilation operators — one has  $\langle \alpha | (\hat{a}^\dagger)^k (\hat{a})^\ell | \alpha \rangle = (\alpha^*)^k \alpha^\ell$ , simply because  $\hat{a} | \alpha \rangle = \alpha | \alpha \rangle$  and  $\langle \alpha | \hat{a}^\dagger = \alpha^* \langle \alpha |$ . In particular,  $\langle \alpha | (\hat{n} = \hat{a}^\dagger \hat{a}) | \alpha \rangle = \alpha^* \alpha$ . On the other hand,

$$\hat{n}^2 = \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} = \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} + \hat{a}^\dagger \hat{a} \implies \langle \alpha | \hat{n}^2 | \alpha \rangle = (\alpha^*)^2 \alpha^2 + \alpha^* \alpha = \bar{n}^2 + \bar{n} \quad (\text{S.3})$$

hence  $\Delta n = \sqrt{\langle \hat{n}^2 \rangle - \bar{n}^2} = \sqrt{\bar{n}}$ .

In a similar manner,

$$\hat{q} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger), \quad \hat{q}^2 = \frac{\hbar}{2m\omega} \left( (\hat{a})^2 + (\hat{a}^\dagger)^2 + 2\hat{a}^\dagger \hat{a} + 1 \right) \implies$$

$$\langle \alpha | \hat{q}^2 | \alpha \rangle = \frac{\hbar}{2m\omega} ((\alpha + \alpha^*)^2 + 1) = \langle \alpha | \hat{q} | \alpha \rangle^2 + \frac{\hbar}{2m\omega}$$

and likewise

$$\langle \alpha | \hat{p}^2 | \alpha \rangle = \frac{m\omega\hbar}{2} ((-i\alpha + i\alpha^*)^2 + 1) = \langle \alpha | \hat{p} | \alpha \rangle^2 + \frac{m\omega\hbar}{2},$$

thus

$$\Delta q = \sqrt{\frac{\hbar}{2m\omega}}, \quad \Delta p = \sqrt{\frac{m\omega\hbar}{2}}, \quad \Delta q \Delta p = \frac{\hbar}{2}. \quad (\text{S.4})$$

*Q.E.D.*

Problem 1(c):

In the Schrödinger picture,  $\hat{a}^\dagger$  is time independent, hence  $(d/dt)e^{\alpha\hat{a}^\dagger} = (d\alpha/dt)\hat{a}^\dagger e^{\alpha\hat{a}^\dagger}$ . Using time independence of the magnitude  $|\alpha|$ , we then have

$$\frac{d}{dt} \left( |\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^\dagger} |0\rangle \right) = \frac{d\alpha}{dt} \hat{a}^\dagger |\alpha\rangle = \frac{1}{\alpha} \frac{d\alpha}{dt} \hat{a}^\dagger \hat{a} |\alpha\rangle. \quad (\text{S.5})$$

Specifically, for  $\alpha = \alpha_0 e^{-i\omega t}$ ,  $(d/dt)|\alpha\rangle = -i\omega\hat{a}^\dagger\hat{a}|\alpha\rangle$  and consequently, for the state vector (2),

$$i\hbar \frac{d}{dt} |\psi\rangle = \hbar\omega\hat{a}^\dagger\hat{a} |\psi\rangle + \frac{1}{2}\hbar\omega\psi = \hat{H} |\psi\rangle. \quad (\text{S.6})$$

*Q.E.D.*

Problem 1(d):

First, quantum overlap with the ground state,  $\langle 0 | \alpha \rangle = e^{-|\alpha|^2/2} \langle 0 | e^{\alpha\hat{a}^\dagger} | 0 \rangle = e^{-|\alpha|^2/2}$  because  $\langle 0 | e^{\alpha\hat{a}^\dagger} = \langle 0 |$  (note  $\langle 0 | \hat{a}^\dagger = 0$ ). Now, for two different coherent states,

$$\langle \alpha | \beta \rangle = e^{-|\alpha|^2/2} \langle 0 | e^{\alpha^*\hat{a}} | \beta \rangle = e^{-|\alpha|^2/2} \langle 0 | e^{\alpha^*\beta} | \beta \rangle = e^{-|\alpha|^2/2} e^{\alpha^*\beta} e^{-|\beta|^2/2}$$

or in terms of the probability overlap,

$$|\langle \alpha | \beta \rangle|^2 = e^{-|\alpha-\beta|^2}. \quad (\text{S.7})$$

Problem 1(e):

Generalization of coherent states to multi-oscillatory systems and further to the creation / annihilation fields is completely straightforward:

$$|\text{coherent}\rangle \stackrel{\text{def}}{=} \exp(\hat{F}^\dagger - \hat{F}) |0\rangle = e^{-\bar{N}/2} e^{\hat{F}^\dagger} |0\rangle \quad (\text{S.8})$$

where

$$\hat{F}^\dagger = \alpha \hat{a}^\dagger \rightarrow \sum_i \alpha_i \hat{a}_i^\dagger \rightarrow \int d^3 \mathbf{x} \Phi(\mathbf{x}) \hat{\Psi}^\dagger(\mathbf{x}). \quad (\text{S.9})$$

Similar to the single-oscillator theory,  $(\hat{\Psi}(\mathbf{x}) - \Phi(\mathbf{x}))e^{\hat{F}^\dagger} = e^{\hat{F}^\dagger} \hat{\Psi}^\dagger(\mathbf{x})$ , hence eq. (3).

Problem 1(f):

Again, using eq. (3) and its hermitian conjugate, we have

$$\langle \Phi | \hat{\Psi}^\dagger(\mathbf{x}_1) \cdots \hat{\Psi}^\dagger(\mathbf{x}_k) \hat{\Psi}(\mathbf{y}_1) \cdots \hat{\Psi}(\mathbf{y}_\ell) | \Phi \rangle = \Phi^*(\mathbf{x}_1) \cdots \Phi^*(\mathbf{x}_k) \Phi(\mathbf{y}_1) \cdots \Phi(\mathbf{y}_\ell) \quad (\text{S.10})$$

for any *normal-ordered* product of the quantum fields. Specifically, for the particle-number operator  $\hat{N}$  we have eq. (4), while for its square — whose normal-ordered form

$$\hat{N}^2 = \iint d^3 \mathbf{x} d^3 \mathbf{y} \hat{\Psi}^\dagger(\mathbf{x}) \hat{\Psi}^\dagger(\mathbf{y}) \hat{\Psi}(\mathbf{x}) \hat{\Psi}(\mathbf{y}) + \int d^3 \mathbf{x} \hat{\Psi}^\dagger(\mathbf{x}) \hat{\Psi}(\mathbf{x}) \quad (\text{S.11})$$

generalizes eq. (S.3) — we have

$$\langle \Phi | \hat{N}^2 | \Phi \rangle = \iint d^3 \mathbf{x} d^3 \mathbf{y} \Phi^*(\mathbf{x}) \Phi^*(\mathbf{y}) \Phi(\mathbf{x}) \Phi(\mathbf{y}) + \int d^3 \mathbf{x} \Phi^*(\mathbf{x}) \Phi(\mathbf{x}) = \langle \Phi | \hat{N} | \Phi \rangle^2 + \langle \Phi | \hat{N} | \Phi \rangle \quad (\text{S.12})$$

and hence  $\Delta N = \sqrt{\bar{N}}$ , *Q.E.D.*

Problem 1(g):

First of all, if  $\Phi(\mathbf{x}, t)$  satisfies the classical field equation — which looks exactly like a one-particle Schrödinger equation — then  $\bar{N}$  remains constant. (This is undergraduate-level QM.)

Also, in the Schrödinger picture of the QFT,

$$\frac{d}{dt} e^{\hat{F}^\dagger} = \frac{d\hat{F}^\dagger}{dt} e^{\hat{F}^\dagger} = \left[ \int d^3\mathbf{x} \frac{\partial\Phi(\mathbf{x}, t)}{\partial t} \hat{\Psi}^\dagger(\mathbf{x}) \right] e^{\hat{F}^\dagger} \quad (\text{S.13})$$

thanks to mutual commutativity of the creation fields. Consequently, exactly as in question (c),

$$\begin{aligned} i\hbar \frac{d}{dt} \left( |\Phi\rangle = e^{-\bar{N}/2} e^{\hat{F}^\dagger} |0\rangle \right) &= \left[ \int d^3\mathbf{x} i\hbar \frac{\partial\Phi(\mathbf{x}, t)}{\partial t} \hat{\Psi}^\dagger(\mathbf{x}) \right] |\Phi\rangle \\ \langle\langle \text{using the classical field equation for } \Phi \rangle\rangle & \\ &= \left[ \int d^3\mathbf{x} \left( \frac{-\hbar^2}{2M} \nabla^2 + V(\mathbf{x}) \right) \Phi(\mathbf{x}) \hat{\Psi}^\dagger(\mathbf{x}) \right] |\Phi\rangle \quad (\text{S.14}) \\ \langle\langle \text{using coherence} \rangle\rangle & \\ &= \left[ \int d^3\mathbf{x} \hat{\Psi}^\dagger(\mathbf{x}) \left( \frac{-\hbar^2}{2M} \nabla^2 + V(\mathbf{x}) \right) \hat{\Psi}(\mathbf{x}) \right] |\Phi\rangle \\ &= \hat{H} |\Phi\rangle. \end{aligned}$$

*Q.E.D.*

Problem 1(h):

Generalizing (d) to multi-oscillatory systems is completely straightforward:

$$|\langle\alpha|\beta\rangle|^2 = \prod_i e^{-|\alpha_i - \beta_i|^2} = \exp\left(-\sum_i |\alpha_i - \beta_i|^2\right)$$

or for the field theory,

$$|\langle\Phi_1|\Phi_2\rangle|^2 = \exp\left(\int d^3\mathbf{x} |\Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})|^2\right), \quad (\text{S.15})$$

which is exponentially small for any macroscopic  $\delta\Phi(\mathbf{x})$ . Indeed a *macroscopic* difference between two coherent states means that  $\delta\Phi$  affects a large number of particles,  $\int |\delta\Phi|^2 \gg 1$ .

Problem 2(a):

Classical equation of motion for a complex field follow from zero variational derivative of the action with respect to both real and imaginary parts if  $\delta\Phi(x)$ . Equivalently, we treat  $\Phi(x)$  and  $\Phi^*(x)$  as independent, thus

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} = \partial^\mu\Phi^*, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)} = \partial^\mu\Phi, \quad \frac{\partial\mathcal{L}}{\partial(\Phi)} = -m^2\Phi^* - \frac{\lambda}{2}(\Phi^*)^2\Phi, \quad \frac{\partial\mathcal{L}}{\partial(\Phi^*)} = -m^2\Phi - \frac{\lambda}{2}\Phi^*\Phi^2,$$

and the Euler–Lagrange field equations

$$\partial^2\Phi^* + m^2\Phi^* + \frac{\lambda}{2}(\Phi^*)^2\Phi = 0, \quad \partial^2\Phi + m^2\Phi + \frac{\lambda}{2}\Phi^*\Phi^2 = 0. \quad (\text{S.16})$$

Consequently, the divergence of the current (7) evaluates to

$$\begin{aligned} \partial_\mu J^\mu &= i\partial_\mu(\Phi^*(\partial^\mu\Phi) - (\partial^\mu\Phi^*)\Phi) = i\Phi^*(\partial^2\Phi) - i(\partial^2\Phi^*)\Phi \\ &= -i\Phi^*(m^2\Phi + \frac{\lambda}{2}\Phi^*\Phi^2) + i(m^2\Phi^* + \frac{\lambda}{2}(\Phi^*)^2\Phi)\Phi = 0. \end{aligned} \quad (\text{S.17})$$

*Q.E.D.*

Problem 2(b):

In the Hamiltonian formalism for the classical fields  $\Phi(x)$  and  $\Phi^*(x)$ , the canonical conjugate fields are

$$\Pi(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\Phi)} = \partial_0\Phi^*(x) \quad \text{and} \quad \Pi^*(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\Phi^*)} = \partial_0\Phi(x). \quad (\text{S.18})$$

The canonical conjugation implies canonical Poisson brackets between the classical fields  $\Phi$ ,  $\Phi^*$ ,  $\Pi$  and  $\Pi^*$  and hence the canonical commutation relation between their quantum counterparts:

$$\begin{aligned} [\hat{\Phi}(\mathbf{x}), \hat{\Phi}(\mathbf{x}')] &= [\hat{\Phi}(\mathbf{x}), \hat{\Phi}^\dagger(\mathbf{x}')] = [\hat{\Phi}^\dagger(\mathbf{x}), \hat{\Phi}^\dagger(\mathbf{x}')] = 0, \\ [\hat{\Pi}(\mathbf{x}), \hat{\Pi}(\mathbf{x}')] &= [\hat{\Pi}(\mathbf{x}), \hat{\Pi}^\dagger(\mathbf{x}')] = [\hat{\Pi}^\dagger(\mathbf{x}), \hat{\Pi}^\dagger(\mathbf{x}')] = 0, \\ [\hat{\Phi}(\mathbf{x}), \hat{\Pi}^\dagger(\mathbf{x}')] &= [\hat{\Phi}^\dagger(\mathbf{x}), \hat{\Pi}(\mathbf{x}')] = 0, \end{aligned} \quad (\text{S.19})$$

$$[\hat{\Phi}(\mathbf{x}), \hat{\Pi}(\mathbf{x}')] = [\hat{\Phi}^\dagger(\mathbf{x}), \hat{\Pi}^\dagger(\mathbf{x}')] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}').$$

(In the Heisenberg picture for the quantum fields, these commutation relations hold for equal times  $t = t'$  only.) Classically, the Hamiltonian density is

$$\begin{aligned} \mathcal{H} &= \Pi\partial_0\Phi + \Pi^*\partial_0\Phi^* - \mathcal{L} \\ &= \Pi^*\Pi + \nabla\Phi^* \cdot \nabla\Phi + m^2\Phi^*\Phi + \frac{1}{4}\lambda(\Phi^*\Phi)^2, \end{aligned} \quad (\text{S.20})$$

so the quantum theory's Hamiltonian is obviously (8) (modulo operator-ordering ambiguity).  
*Q.E.D.*

Problem 2(c):

Fourier transforming the canonical commutation relations (S.19) results in

$$\begin{aligned} [\hat{\Phi}_{\mathbf{p}}, \hat{\Phi}_{\mathbf{p}'}] &= [\hat{\Phi}_{\mathbf{p}}, \hat{\Phi}_{\mathbf{p}'}^\dagger] = [\hat{\Phi}_{\mathbf{p}}^\dagger, \hat{\Phi}_{\mathbf{p}'}^\dagger] = 0, \\ [\hat{\Pi}_{\mathbf{p}}, \hat{\Pi}_{\mathbf{p}'}] &= [\hat{\Pi}_{\mathbf{p}}, \hat{\Pi}_{\mathbf{p}'}^\dagger] = [\hat{\Pi}_{\mathbf{p}}^\dagger, \hat{\Pi}_{\mathbf{p}'}^\dagger] = 0, \\ [\hat{\Phi}_{\mathbf{p}}, \hat{\Pi}_{\mathbf{p}'}^\dagger] &= [\hat{\Phi}_{\mathbf{p}}^\dagger, \hat{\Pi}_{\mathbf{p}'}] = 0, \\ [\hat{\Phi}_{\mathbf{p}}, \hat{\Pi}_{\mathbf{p}'}] &= [\hat{\Phi}_{\mathbf{p}}^\dagger, \hat{\Pi}_{\mathbf{p}'}^\dagger] = (2\pi)^3 i\delta^{(3)}(\mathbf{p} + \mathbf{p}'). \end{aligned} \quad (\text{S.21})$$

Consequently,

$$[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{p}'}] = [\hat{a}_{\mathbf{p}}, \hat{b}_{\mathbf{p}'}^\dagger] = [\hat{b}_{\mathbf{p}}^\dagger, \hat{b}_{\mathbf{p}'}^\dagger] = 0 \quad (\text{S.22})$$

because all  $\Phi_{\mathbf{p}}$  and all  $\Pi_{\mathbf{p}}^\dagger$  commute with each other. Similarly,

$$[\hat{b}_{\mathbf{p}}, \hat{b}_{\mathbf{p}'}] = [\hat{b}_{\mathbf{p}}, \hat{a}_{\mathbf{p}'}^\dagger] = [\hat{a}_{\mathbf{p}}^\dagger, \hat{a}_{\mathbf{p}'}^\dagger] = 0 \quad (\text{S.23})$$

because all  $\Phi_{\mathbf{p}}^\dagger$  and all  $\Pi_{\mathbf{p}}$  commute with each other too. Less obviously

$$[\hat{a}_{\mathbf{p}}, \hat{b}_{\mathbf{p}'}] = \frac{i}{2}\sqrt{\frac{E_{\mathbf{p}}}{E_{\mathbf{p}'}}} (2\pi)^3 i\delta^{(3)}(\mathbf{p} + \mathbf{p}') + \frac{i}{2}\sqrt{\frac{E_{\mathbf{p}'}}{E_{\mathbf{p}}}} (2\pi)^3 (-i)\delta^{(3)}(-\mathbf{p} - \mathbf{p}') = 0 \quad (\text{S.24})$$

and likewise  $[\hat{a}_{\mathbf{p}}^\dagger, \hat{b}_{\mathbf{p}'}^\dagger] = 0$ . Finally,

$$\begin{aligned} [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{p}'}^\dagger] &= [\hat{b}_{\mathbf{p}}, \hat{b}_{\mathbf{p}'}^\dagger] = \frac{-i}{2} \sqrt{\frac{E_{\mathbf{p}}}{E_{\mathbf{p}'}}} (2\pi)^3 i \delta^{(3)}(\mathbf{p} - \mathbf{p}') + \frac{i}{2} \sqrt{\frac{E_{\mathbf{p}'}}{E_{\mathbf{p}}}} (2\pi)^3 (-i) \delta^{(3)}(\mathbf{p}' - \mathbf{p}) \\ &= (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}'). \end{aligned} \quad (\text{S.25})$$

*Q.E.D.*

Problem 2(d):

First, Fourier-transforming the free Hamiltonian gives us

$$\hat{H}_{\text{free}} = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \left( \hat{\Pi}_{\mathbf{p}}^\dagger \hat{\Pi}_{\mathbf{p}} + E_{\mathbf{p}}^2 \hat{\Phi}_{\mathbf{p}}^\dagger \hat{\Phi}_{\mathbf{p}} \right). \quad (\text{S.26})$$

Second, we reverse the definitions (9) to obtain

$$\hat{\Phi}_{\mathbf{p}} = \frac{\hat{a}_{\mathbf{p}} + \hat{b}_{-\mathbf{p}}^\dagger}{\sqrt{2E_{\mathbf{p}}}} \quad \text{and} \quad \hat{\Pi}_{\mathbf{p}} = \sqrt{\frac{1}{2}E_{\mathbf{p}}} \left( -i\hat{b}_{\mathbf{p}} + i\hat{a}_{-\mathbf{p}}^\dagger \right). \quad (\text{S.27})$$

Third, we calculate

$$\begin{aligned} E_{\mathbf{p}}^2 \hat{\Phi}_{\mathbf{p}}^\dagger \hat{\Phi}_{\mathbf{p}} + \hat{\Pi}_{-\mathbf{p}} \hat{\Pi}_{-\mathbf{p}}^\dagger &= \frac{1}{2} E_{\mathbf{p}} (\hat{a}_{\mathbf{p}}^\dagger + \hat{b}_{-\mathbf{p}}) (\hat{a}_{\mathbf{p}} + \hat{b}_{-\mathbf{p}}^\dagger) + \frac{1}{2} E_{\mathbf{p}} (i\hat{a}_{\mathbf{p}}^\dagger - i\hat{b}_{-\mathbf{p}}) (-i\hat{a}_{\mathbf{p}} + i\hat{b}_{-\mathbf{p}}^\dagger) \\ &= E_{\mathbf{p}} (\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \hat{b}_{-\mathbf{p}} \hat{b}_{-\mathbf{p}}^\dagger). \end{aligned} \quad (\text{S.28})$$

Finally, we put eqs. (S.28) and (S.26) together and derive

$$\begin{aligned} \hat{H}_{\text{free}} &= \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \left( E_{\mathbf{p}}^2 \hat{\Phi}_{\mathbf{p}}^\dagger \hat{\Phi}_{\mathbf{p}} + \hat{\Pi}_{-\mathbf{p}} \hat{\Pi}_{-\mathbf{p}}^\dagger \right) \\ &= \int \frac{d^3 \mathbf{p}}{(2\pi)^3} E_{\mathbf{p}} (\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \hat{b}_{-\mathbf{p}} \hat{b}_{-\mathbf{p}}^\dagger) \\ &= \int \frac{d^3 \mathbf{p}}{(2\pi)^3} E_{\mathbf{p}} (\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}}) + \text{const.} \end{aligned} \quad (\text{S.29})$$

*Q.E.D.*

Problem 2(e):

Identifying the current (7) as the electric current, we have classical charge density

$$J^0 = i\Phi^*(\partial^0\Phi) - i(\partial^0\Phi^*)\Phi = i\Phi^*\Pi^* - i\Phi\Pi, \quad (\text{S.30})$$

which in a quantum theory gives us the electric charge operator

$$\hat{Q} = \int d^3\mathbf{x} \hat{J}^0(\mathbf{x}) = \int d^3\mathbf{x} \left( \frac{i}{2}\{\hat{\Phi}^\dagger, \hat{\Pi}^\dagger\} - \frac{i}{2}\{\hat{\Phi}, \hat{\Pi}\} + \text{const} \right), \quad (\text{S.31})$$

where the unknown constant reflects the ambiguity of operator ordering in quantizing a classical quantity. We shall see momentarily that setting this constant to zero results in the normal ordering of  $\hat{Q}$  in terms of creation and annihilation operators and hence in zero electric charge of the vacuum. Physically, this is what we want — hence, the first eq. (12).

Our next step is to Fourier transforming eq. (S.31), which yields

$$\hat{Q} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left( \frac{i}{2}\{\hat{\Phi}_{\mathbf{p}}^\dagger, \hat{\Pi}_{-\mathbf{p}}^\dagger\} - \frac{i}{2}\{\hat{\Phi}_{\mathbf{p}}, \hat{\Pi}_{-\mathbf{p}}\} \right). \quad (\text{S.32})$$

Furthermore, simple algebra gives

$$\begin{aligned} -i\{\hat{\Phi}_{\mathbf{p}}, \hat{\Pi}_{-\mathbf{p}}\} &= \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} - \hat{b}_{-\mathbf{p}}^\dagger \hat{b}_{-\mathbf{p}} + \hat{a}_{\mathbf{p}}^\dagger \hat{b}_{-\mathbf{p}}^\dagger - \hat{a}_{\mathbf{p}} \hat{b}_{-\mathbf{p}}, \\ +i\{\hat{\Phi}_{\mathbf{p}}^\dagger, \hat{\Pi}_{-\mathbf{p}}^\dagger\} &= \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} - \hat{b}_{-\mathbf{p}}^\dagger \hat{b}_{-\mathbf{p}} - \hat{a}_{\mathbf{p}}^\dagger \hat{b}_{-\mathbf{p}}^\dagger + \hat{a}_{\mathbf{p}} \hat{b}_{-\mathbf{p}}, \end{aligned} \quad (\text{S.33})$$

(*cf.* eq. (S.27)) and therefore,

$$\hat{Q} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left( \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} - \hat{b}_{-\mathbf{p}}^\dagger \hat{b}_{-\mathbf{p}} \right) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left( \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} - \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}} \right). \quad (\text{S.34})$$

*Q.E.D.*

Problem 2(f):

According to eq. (13),

$$T^{0i} = \partial^0 \Phi^* \partial^i \Phi + \partial^0 \Phi \partial^i \Phi^* = \Pi \partial^i \Phi + \Pi^* \partial^i \Phi^* \quad (\text{S.35})$$

and hence

$$\hat{\mathbf{P}}_{\text{mech}} = \int d^3 \mathbf{x} \left( \frac{1}{2} \{ \hat{\Pi}, -\nabla \hat{\Phi} \} + \frac{1}{2} \{ \hat{\Pi}^\dagger, -\nabla \hat{\Phi}^\dagger \} \right). \quad (\text{S.36})$$

Let us Fourier transform

$$\hat{\mathbf{P}}_{\text{mech}} = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \left( \frac{i}{2} \mathbf{p} \{ \hat{\Phi}_{\mathbf{p}}^\dagger, \hat{\Pi}_{-\mathbf{p}}^\dagger \} - \frac{i}{2} \mathbf{p} \{ \hat{\Phi}_{\mathbf{p}}, \hat{\Pi}_{-\mathbf{p}} \} \right) \quad (\text{S.37})$$

and then make use of eqs. (S.33), which yield

$$\hat{\mathbf{P}}_{\text{mech}} = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \left( \mathbf{p} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} - \mathbf{p} \hat{b}_{-\mathbf{p}}^\dagger \hat{b}_{-\mathbf{p}} \right) = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \left( \mathbf{p} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \mathbf{p} \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}} \right). \quad (\text{S.38})$$

*Q.E.D.*