Problem $\mathbf{1}$(a):

The conjugacy relations $\hat{A}^\dagger_{\mathbf{k},\lambda} = -\hat{A}_{-\mathbf{k},\lambda}$, $\hat{E}^\dagger_{\mathbf{k},\lambda} = -\hat{E}_{-\mathbf{k},\lambda}$ follow from hermiticity of the $\hat{\mathbf{A}}(\mathbf{x})$ and $\hat{\mathbf{E}}(\mathbf{x})$ quantum fields and from the third eq. (6) for the polarization vectors:

$$\hat{A}^\dagger_{\mathbf{k},\lambda} \;=\; \int\! d^3\mathbf{x}\, e^{+i\mathbf{k}\mathbf{x}} \mathbf{e}_\lambda(\mathbf{k})\cdot\hat{\mathbf{A}}^\dagger(\mathbf{x}) \;=\; \int\! d^3\mathbf{x}\, e^{-i(-\mathbf{k})\mathbf{x}}(-\mathbf{e}^*_\lambda(-\mathbf{k}))\cdot\hat{\mathbf{A}} \;=\; -\hat{A}_{-\mathbf{k},\lambda}\,, \tag{S.1}$$

and ditto for the $\hat{E}^\dagger_{\mathbf{k},\lambda} = -\hat{E}_{-\mathbf{k},\lambda}$. The equal-time commutation relations follow from eqs. (1): Obviously,

$$[\hat{A}_{\mathbf{k},\lambda}, \hat{A}_{\mathbf{k}',\lambda'}] \;=\; 0, \qquad [\hat{E}_{\mathbf{k},\lambda}, \hat{E}_{\mathbf{k}',\lambda'}] \;=\; 0. \tag{S.2}$$

Less obviously,

$$\begin{aligned}
[\hat{A}_{\mathbf{k},\lambda}, \hat{E}^\dagger_{\mathbf{k}',\lambda'}] \;&=\; \int\! d^3\mathbf{x}\, e^{-i\mathbf{k}\mathbf{x}}\big(\mathbf{e}^*_\lambda(\mathbf{k})\big)^i \int\! d^3\mathbf{y}\, e^{+i\mathbf{k}'\mathbf{y}}\big(\mathbf{e}_{\lambda'}(\mathbf{k}')\big)^j\, [\hat{A}^i(\mathbf{x}), \hat{E}^j(\mathbf{y})] \\
&=\; \int\! d^3\mathbf{x}\, e^{-i(\mathbf{k}-\mathbf{k}')\mathbf{x}}\,\big(-i\mathbf{e}^*_\lambda(\mathbf{k})\cdot\mathbf{e}_{\lambda'}(\mathbf{k}')\big) \\
&=\; -i(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\lambda,\lambda'}\,,
\end{aligned} \tag{S.3}$$

or equivalently,

$$[\hat{A}_{\mathbf{k},\lambda}, \hat{E}_{\mathbf{k}',\lambda'}] \;=\; +i(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\delta_{\lambda,\lambda'}\,. \tag{S.4}$$

Problem $\mathbf{1}$(b):

There are four terms in the Hamiltonian density (2), so let us consider them one by one. Combining Fourier transform with decomposition into polarization modes it is easy to see that in light of eq. (4),

$$\int\! d^3\mathbf{x}\,\hat{\mathbf{E}}^2(\mathbf{x}) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda \hat{E}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda} \tag{S.5}$$

and likewise

$$\int\! d^3\mathbf{x}\,\hat{\mathbf{A}}^2(\mathbf{x}) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda \hat{A}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda}\,. \tag{S.6}$$

Furthermore, using eq. (5) we have

$$\nabla\times\hat{\mathbf{A}}(\mathbf{x}) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda e^{i\mathbf{k}\mathbf{x}}\Big(i\mathbf{k}\times\mathbf{e}_\lambda(\mathbf{k})=\lambda|\mathbf{k}|\mathbf{e}_\lambda(\mathbf{k})\Big)\,\hat{A}_{\mathbf{k},\lambda} \tag{S.7}$$

and hence

$$\int\! d^3\mathbf{x}\,\Big(\nabla\times\hat{\mathbf{A}}(\mathbf{x})\Big)^2 \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda \lambda^2\mathbf{k}^2\,\hat{A}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda}\,. \tag{S.8}$$

Finally, the first eq. (6) gives us

$$\nabla\cdot\hat{\mathbf{E}}(\mathbf{x}) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda e^{i\mathbf{k}\mathbf{x}}\Big(i\mathbf{k}\cdot\mathbf{e}_\lambda(\mathbf{k})=i|\mathbf{k}|\delta_{\lambda,0}\Big)\,\hat{E}_{\mathbf{k},\lambda} \tag{S.9}$$

and hence

$$\int\! d^3\mathbf{x}\,\Big(\nabla\cdot\hat{\mathbf{E}}(\mathbf{x})\Big)^2 \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\mathbf{k}^2\hat{E}^\dagger_{\mathbf{k},0}\hat{E}_{\mathbf{k},0}\,. \tag{S.10}$$

In light of all these formulæ, we assemble the Hamiltonian (2) as

$$\hat{H} \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda\left(\left(\frac{1}{2}+\frac{\mathbf{k}^2}{2m^2}\delta_{\lambda,0}=\frac{C_{\mathbf{k},\lambda}}{2}\right)\hat{E}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda} \;+\; \left(\frac{m^2+\lambda^2\mathbf{k}^2}{2}=\frac{\omega^2_{\mathbf{k}}}{2C_{\mathbf{k},\lambda}}\right)\hat{A}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda}\right). \tag{8}$$

Problem **1**(c):

Given eqs. (S.2) and (S.4), we have

$$\begin{aligned}
[\hat{a}_{\mathbf{k},\lambda},\hat{a}_{\mathbf{k}',\lambda'}] \;&=\; \frac{-i}{2}\sqrt{\frac{\omega_{\mathbf{k}}C_{\mathbf{k}',\lambda'}}{\omega_{\mathbf{k}'}C_{\mathbf{k},\lambda}}}\,\Big([\hat{A}_{\mathbf{k},\lambda},\hat{E}_{\mathbf{k}',\lambda'}]=(+i)(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\delta_{\lambda,\lambda'}\Big)\\
&\quad+\;\frac{-i}{2}\sqrt{\frac{\omega_{\mathbf{k}'}C_{\mathbf{k},\lambda}}{\omega_{\mathbf{k}}C_{\mathbf{k}',\lambda'}}}\,\Big([\hat{E}_{\mathbf{k},\lambda},\hat{A}_{\mathbf{k}',\lambda'}]=(-i)(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\delta_{\lambda,\lambda'}\Big)\\
&=\; \tfrac{1}{2}(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\delta_{\lambda,\lambda'} \;-\; \tfrac{1}{2}(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}')\delta_{\lambda,\lambda'} \;=\; 0
\end{aligned} \tag{S.11}$$

and likewise $[\hat{a}^\dagger_{\mathbf{k},\lambda},\hat{a}^\dagger_{\mathbf{k}',\lambda'}]=0$. On the other hand,

$$\begin{aligned}
[\hat{a}_{\mathbf{k},\lambda},\hat{a}^\dagger_{\mathbf{k}',\lambda'}] &= \frac{+i}{2}\sqrt{\frac{\omega_{\mathbf{k}}C_{\mathbf{k}',\lambda'}}{\omega_{\mathbf{k}'}C_{\mathbf{k},\lambda}}}\left([\hat{A}_{\mathbf{k},\lambda},\hat{E}^\dagger_{\mathbf{k}',\lambda'}]=(-i)(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\lambda,\lambda'}\right)\\
&\quad+\ \frac{-i}{2}\sqrt{\frac{\omega_{\mathbf{k}'}C_{\mathbf{k},\lambda}}{\omega_{\mathbf{k}}C_{\mathbf{k}',\lambda'}}}\left([\hat{E}_{\mathbf{k},\lambda},\hat{A}^\dagger_{\mathbf{k}',\lambda'}]=(+i)(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\lambda,\lambda'}\right)\\
&= \tfrac{1}{2}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\lambda,\lambda'}\ +\ \tfrac{1}{2}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\lambda,\lambda'}\\
&= (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\lambda,\lambda'}\,.
\end{aligned}\tag{S.12}$$

$\mathcal{Q.E.D.}$

<u>Problem **1**(d)</u>:

Expanding the operators $\hat{a}_{\mathbf{k},\lambda}$ and $\hat{a}^\dagger_{\mathbf{k},\lambda}$ according to eqs. (9), we have

$$\omega_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}\ =\ \frac{\omega^2_{\mathbf{k}}}{2C_{\mathbf{k},\lambda}}\,\hat{A}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda}\ +\ \frac{C_{\mathbf{k},\lambda}}{2}\,\hat{E}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda}\ +\ \frac{i\omega_{\mathbf{k}}}{2}\,\hat{E}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda}\ -\ \frac{i\omega_{\mathbf{k}}}{2}\,\hat{A}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda}\tag{S.13}$$

Upon integrating over momentum $\mathbf{k}$ and summing over polarizations as in eq. (10), the first two terms on the right hand side of eq. (S.13) reproduce the Hamiltonian (8), so we need to show that the remaining two terms integrate to a constant. The trick here is to change the integration variable $\mathbf{k}\to-\mathbf{k}$ in the last term,

$$\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda\frac{i\omega_{\mathbf{k}}}{2}\,\hat{A}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda}\ =\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda\frac{i\omega_{\mathbf{k}}}{2}\,\hat{A}^\dagger_{-\mathbf{k},\lambda}\hat{E}_{-\mathbf{k},\lambda}\ =\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda\frac{i\omega_{\mathbf{k}}}{2}\,\hat{A}_{+\mathbf{k},\lambda}\hat{E}^\dagger_{+\mathbf{k},\lambda}\,.\tag{S.14}$$

Consequently,

$$\begin{aligned}
\hat{H}\ -\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda\omega_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}\ &=\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda\frac{i\omega_{\mathbf{k}}}{2}\left(-\hat{E}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda}\ +\ \hat{A}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda}\right)\\
&=\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda\frac{i\omega_{\mathbf{k}}}{2}\,[\hat{A}_{\mathbf{k},\lambda},\hat{E}^\dagger_{\mathbf{k},\lambda}]\\
&=\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda\frac{\omega_{\mathbf{k}}}{2}\,(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}=\mathbf{0})\\
&\equiv E_{\mathrm{vacuum}}
\end{aligned}\tag{S.15}$$

which is indeed a c-number constant, albeit divergent. $\mathcal{Q.E.D.}$

Physically, the vacuum energy (S.15) is the net zero-point energy of all the oscillatory modes of the vector field theory. This energy is infinite for two reasons, one having do do with the infinite volume of space and the other with its perfect continuity. The infinite-volume divergence of $\int d^3\mathbf{x}$ of a constant vacuum energy *density* manifest itself via the $(2\pi)^3\delta^{(3)}(\mathbf{0})$ factor, which is simply the Fourier transform of $\int d^3\mathbf{x}(1)$. Indeed, had we quantized the theory in a very large but finite box, we would have obtained the L^3 volume factor in eq. (S.15) instead of the delta function. In other words, the vacuum has energy density

$$\frac{\text{Energy}}{\text{Volume}}\bigg|_{\text{Vacuum}} = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\, 3\times\frac{\omega_{\mathbf{k}}}{2}\,. \tag{S.16}$$

Alas, this integral diverges at large momenta thus endowing the theory with an infinite vacuum energy. This is a generic problem of all Quantum Field Theories in a perfectly continuous space (and hence unlimitedly high momenta). Ultimately, this problem should be resolved by the fundamental theory of physics at ultra-short distances, whatever such theory might be.

Fortunately, for all practical purposes, we may safely disregard any c-number constant term in the Hamiltonian, even if such term is infinite — and that is exactly what we shall do in this course!

<u>Problem **1**(e)</u>:

Reversing eqs. (9), we have

$$\hat{A}_{\mathbf{k},\lambda} = \sqrt{\frac{C_{\mathbf{k},\lambda}}{2\omega_{\mathbf{k}}}}\left(\hat{a}_{\mathbf{k},\lambda} - \hat{a}^\dagger_{-\mathbf{k},\lambda}\right) \tag{S.17}$$

and consequently

$$\begin{aligned}
\hat{\mathbf{A}}(\mathbf{x}) &= \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{e^{i\mathbf{k}\mathbf{x}}}{\sqrt{2\omega_{\mathbf{k}}}}\sum_\lambda\sqrt{C_{\mathbf{k},\lambda}}\,\mathbf{e}_\lambda(\mathbf{k})\left(\hat{a}_{\mathbf{k},\lambda} - \hat{a}^\dagger_{-\mathbf{k},\lambda}\right)\\
&= \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{e^{+i\mathbf{k}\mathbf{x}}}{\sqrt{2\omega_{\mathbf{k}}}}\sum_\lambda\sqrt{C_{\mathbf{k},\lambda}}\,\mathbf{e}_\lambda(\mathbf{k})\,\hat{a}_{\mathbf{k},\lambda} - \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{e^{-i\mathbf{k}\mathbf{x}}}{\sqrt{2\omega_{\mathbf{k}}}}\sum_\lambda\sqrt{C_{\mathbf{k},\lambda}}\,\mathbf{e}_\lambda(-\mathbf{k})\,\hat{a}_{+\mathbf{k},\lambda}\\
&= \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\sum_\lambda\sqrt{C_{\mathbf{k},\lambda}}\left(e^{+i\mathbf{k}\mathbf{x}}\mathbf{e}_\lambda(\mathbf{k})\,\hat{a}_{\mathbf{k},\lambda} + e^{-i\mathbf{k}\mathbf{x}}\mathbf{e}^*_\lambda(\mathbf{k})\,\hat{a}^\dagger_{\mathbf{k},\lambda}\right).
\end{aligned} \tag{S.18}$$

It remains to work out the time dependence in the Heisenberg picture. For the free field governed by Hamiltonian (10), $\hat{a}_{\mathbf{k},\lambda}(t) = e^{-i\omega t}\hat{a}_{\mathbf{k},\lambda}(0)$ and $\hat{a}^\dagger_{\mathbf{k},\lambda}(t) = e^{+i\omega t}\hat{a}^\dagger_{\mathbf{k},\lambda}(0)$ where $\omega \equiv \omega_{\mathbf{k}}$. Substituting this time dependence into eq. (S.18) and switching to relativistic notations, we immediately arrive at

$$\hat{\mathbf{A}}(x) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\sum_\lambda \sqrt{C_{\mathbf{k},\lambda}}\left(e^{-ikx}\mathbf{e}_\lambda(\mathbf{k})\,\hat{a}_{\mathbf{k},\lambda}(0) \;+\; e^{+ikx}\mathbf{e}^*_\lambda(\mathbf{k})\,\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\right)_{k^0=+\omega_{\mathbf{k}}}. \tag{11}$$

$\mathcal{Q.E.D.}$

Problem $\mathbf{1}$(f):

The 3–scalar field $\hat{A}^0(x)$ is governed by eqs. (3) and (S.9),

$$\hat{A}^0(\mathbf{x},t) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{-i|\mathbf{k}|}{m^2}\,e^{i\mathbf{kx}}\,\hat{E}_{\mathbf{k},0}(t). \tag{S.19}$$

Reversing eqs. (9) for the $\hat{E}_{\mathbf{k},\lambda}$ operator, we have

$$\hat{E}_{\mathbf{k},\lambda} \;=\; i\sqrt{\frac{\omega_{\mathbf{k}}}{2C_{\mathbf{k},\lambda}}}\left(\hat{a}_{\mathbf{k},\lambda} + \hat{a}^\dagger_{-\mathbf{k},\lambda}\right). \tag{S.20}$$

In particular,

$$\hat{E}_{\mathbf{k},0} \;=\; \frac{im}{\sqrt{2\omega_{\mathbf{k}}}}\left(\hat{a}_{\mathbf{k},0} + \hat{a}^\dagger_{-\mathbf{k},0}\right)$$

and hence

$$\begin{aligned}
\hat{A}^0(\mathbf{x}) \;&=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{|\mathbf{k}|}{m}\,\frac{e^{+i\mathbf{kx}}}{\sqrt{2\omega_{\mathbf{k}}}}\left(\hat{a}_{\mathbf{k},0} + \hat{a}^\dagger_{-\mathbf{k},0}\right)\\
&=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{|\mathbf{k}|}{m}\,\frac{e^{+i\mathbf{kx}}}{\sqrt{2\omega_{\mathbf{k}}}}\,\hat{a}_{\mathbf{k},0} \;+\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{|\mathbf{k}|}{m}\,\frac{e^{-i\mathbf{kx}}}{\sqrt{2\omega_{\mathbf{k}}}}\,\hat{a}^\dagger_{+\mathbf{k},0}\\
&=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\,\frac{|\mathbf{k}|}{m}\left(e^{+i\mathbf{kx}}\,\hat{a}_{\mathbf{k},0} \;+\; e^{-i\mathbf{kx}}\,\hat{a}^\dagger_{\mathbf{k},0}\right).
\end{aligned} \tag{S.21}$$

As to the time dependence, it works exactly as in eq. (11) for the vector field, thus

$$\hat{A}^0(x) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\,\frac{|\mathbf{k}|}{m}\left(e^{-ikx}\,\hat{a}_{\mathbf{k},0}(0) \;+\; e^{+ikx}\,\hat{a}^\dagger_{\mathbf{k},0}(0)\right)_{k^0=+\omega_{\mathbf{k}}}. \tag{S.22}$$

Similarity between eqs. (11) and (S.22) allows us to combine them into eq. (12) where

$$\mathbf{f}(\mathbf{k},\lambda) \;=\; C_{\mathbf{k},\lambda}\,\mathbf{e}_\lambda(\mathbf{k}) \quad\text{and}\quad f^0(\mathbf{k},\lambda) \;=\; \frac{|\mathbf{k}|}{m}\,\delta_{\lambda,0} \tag{S.23}$$

or equivalently, (13).

Problem **1**(g):

According to eq. (12),

$$(\partial^2+m^2)\hat{A}^\mu(x) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\sum_\lambda\Bigl((-k^2+m^2)e^{-ikx}f^\mu(\mathbf{k},\lambda)\,\hat{a}_{\mathbf{k},\lambda}(0) \\ \qquad +\;(-k^2+m^2)e^{+ikx}f^{*\mu}(\mathbf{k},\lambda)\,\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\Bigr)_{k^0=+\omega_{\mathbf{k}}}, \tag{S.24}$$

which vanishes because $(-k^2+m^2)=0$ for $k^0=\omega_{\mathbf{k}}$. Likewise,

$$\partial_\mu\hat{A}^\mu(x) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\sum_\lambda\Bigl(e^{-ikx}\bigl(ik_\mu f^\mu(\mathbf{k},\lambda)\bigr)\hat{a}_{\mathbf{k},\lambda}(0) \\ \qquad +\;e^{+ikx}\bigl(-ik_\mu f*^\mu(\mathbf{k},\lambda)\bigr)\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\Bigr)_{k^0=+\omega_{\mathbf{k}}}, \tag{S.25}$$

which vanishes because $k_\mu f^\mu(\mathbf{k},\lambda)=0$ for all polarizations λ. $\mathcal{Q.E.D.}$

Problem **2**(a):

The simplest way to prove this lemma is by direct inspection, component by component:

$$\begin{aligned}
\sum_\lambda f^i(\mathbf{k},\lambda)f^{*j}(\mathbf{k},\lambda) \;&=\; \sum_\lambda e^i_\lambda(\mathbf{k})e^{*j}_\lambda(\mathbf{k}) \;+\; \frac{\mathbf{k}^2}{m^2}\,e^i_0(\mathbf{k})e^{*j}_0(\mathbf{k}) \;=\; \delta^{ij} \;+\; \frac{k^ik^j}{m^2}\,;\\
\sum_\lambda f^i(\mathbf{k},\lambda)f^{*0}(\mathbf{k},\lambda) \;&=\; f^i(\mathbf{k},0)f^{*0}(\mathbf{k},0) \;=\; \frac{k^i\omega_{\mathbf{k}}}{m^2}\,;\\
\sum_\lambda f^0(\mathbf{k},\lambda)f^{*0}(\mathbf{k},\lambda) \;&=\; \bigl|f^0(\mathbf{k},0)\bigr|^0 \;=\; \frac{\mathbf{k}^2}{m^2} \;=\; -1\;+\;\frac{\omega^2_{\mathbf{k}}}{m^2}\,.
\end{aligned} \tag{S.26}$$

Alternatively, we may use the fact that the three four-vectors $f^\mu(\mathbf{k},\lambda)$ (fixed $\mathbf{k}$, $\lambda=-1,0,+1$) are orthogonal to each other and also to the $k^\mu=(\omega_{\mathbf{k}},\mathbf{k})$. Furthermore, each $\bigl(f(\mathbf{k},\lambda)\bigr)^2=-1$. Consequently, the symmetric matrix (in Lorentz indices μ,ν) on the left hand side of eq. (14) has to be (minus) the projection matrix onto four-vectors orthogonal to the k^μ, and that is precisely the matrix appearing on the right hand side of eq. (14) (note $k^2=m^2$).

Problem **2**(b):

The operator product $\hat{A}^\mu(x)\hat{A}^\nu(y)$ comprises $\hat{a}\hat{a}$, $\hat{a}^\dagger\hat{a}^\dagger$, $\hat{a}^\dagger\hat{a}$ and $\hat{a}\hat{a}^\dagger$ terms. The first three kinds of terms have zero matrix elements between vacuum states while $\langle 0|\,\hat{a}_{\mathbf{k},\lambda}\hat{a}^\dagger_{\mathbf{k}',\lambda'}\,|0\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\lambda,\lambda'}$. Consequently,

$$\begin{aligned}
\langle 0|\,\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle &= \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\sum_\lambda\left[e^{-ik(x-y)}\,f^\mu(\mathbf{k},\lambda)f^{*\nu}(\mathbf{k},\lambda)\right]_{k^0=+\omega_\mathbf{k}} \\
&= \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\left[\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{m^2}\right)e^{-ik(x-y)}\right]_{k^0=+\omega_\mathbf{k}} \\
&= \left(-g^{\mu\nu}-\frac{1}{m^2}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\right)\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\left[e^{-ik(x-y)}\right]_{k^0=+\omega_\mathbf{k}} \\
&\equiv \left(-g^{\mu\nu}-\frac{1}{m^2}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\right)D(x-y).
\end{aligned} \tag{15}$$

Problem **2**(c):

Actually, eq. (16) is not completely correct; there is a subtlety associated with time-ordering temporal components A^0 of vector fields, so the correct statement is

$$\begin{aligned}
G_F^{\mu\nu} \equiv \langle 0|\,\mathbf{T}^*\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle &= \left(-g^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{m^2}\right)D_F(x-y) \\
&= \int\frac{d^4\mathbf{k}}{(2\pi)^4}\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{m^2}\right)\frac{ie^{-ik(x-y)}}{k^2-m^2+i0}
\end{aligned} \tag{S.27}$$

where

$$\langle 0|\,\mathbf{T}^*\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle = \langle 0|\,\mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle + i\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y). \tag{S.28}$$

For the explanation of the $\mathbf{T}^*$ modification of the time-ordered product, please see *Quantum Field Theory* by Claude Itzykson and Jean–Bernard Zuber.

To derive eq. (S.27), we start with eq. (15) and immediately see that for the un-modified time-ordering,

$$\begin{aligned}
\langle 0|\,\mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle &= \theta(x^0-y^0)\,\langle 0|\,\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle + \theta(y^0-x^0)\,\langle 0|\,\hat{A}^\nu(y)\hat{A}^\mu(x)\,|0\rangle \\
&= \theta(x^0-y^0)\left(-g^{\mu\nu}-\frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)D(x-y) \\
&\quad + \theta(y^0-x^0)\left(-g^{\mu\nu}-\frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)D(y-x)
\end{aligned} \tag{S.29}$$

On the other hand,

$$D_F(x-y) \;=\; \theta(x^0-y^0)D(x-y) \;+\; \theta(y^0-x^0)D(y-x) \tag{S.30}$$

and hence

$$\begin{aligned}\left(-g^{\mu\nu} \;-\; \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right) D_F(x-y) \;&=\; \theta(x^0-y^0)\left(-g^{\mu\nu} \;-\; \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right) D(x-y)\\ &\quad+\; \theta(y^0-x^0)\left(-g^{\mu\nu} \;-\; \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right) D(y-x)\\ &\quad-\; i\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y)\end{aligned} \tag{S.31}$$

where the δ-function term arises from taking time derivatives of the θ-functions. (*cf.* explanation of $(\partial^2+m^2)D_F(x-y) = -i\delta^{(4)}(x-y)$ in class.) Comparing eqs. (S.29) and (S.31), we obtain

$$\langle 0|\,\mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \;=\; \left(-g^{\mu\nu} \;-\; \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right) D_F(x-y) \;-\; i\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y) \tag{S.32}$$

and hence the first line of eq. (S.27). The second line follows from the momentum-space form of the scalar propagator

$$\begin{aligned}\left(-g^{\mu\nu} \;-\; \frac{1}{m^2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\right)\left[D_F(x-y) \;=\int\frac{d^4\mathbf{k}}{(2\pi)^4}\,\frac{ie^{-ik(x-y)}}{k^2-m^2+i0}\right]\\ =\int\frac{d^4\mathbf{k}}{(2\pi)^4}\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{m^2}\right)\frac{ie^{-ik(x-y)}}{k^2-m^2+i0}\,.\end{aligned} \tag{S.33}$$

$\mathcal{Q.E.D.}$

Problem **3**(a):

In terms of the shifted fields, $\widetilde{\Psi}$ and $\widetilde{\Psi}^\dagger$,

$$\begin{aligned}\nabla\Psi^\dagger\cdot\nabla\Psi \;&=\; \nabla\widetilde{\Psi}^\dagger\cdot\nabla\widetilde{\Psi}\\ \mu\,\Psi^\dagger\Psi \;&=\; \mu n \;+\; \mu\sqrt{n}\left(\widetilde{\Psi}^\dagger+\widetilde{\Psi}\right) \;+\; \mu\,\widetilde{\Psi}^\dagger\widetilde{\Psi}\\ \tfrac{1}{2}\lambda\,\Psi^\dagger\Psi^\dagger\Psi\Psi \;&=\; \tfrac{1}{2}\lambda n^2 \;+\; \lambda n\sqrt{n}\left(\widetilde{\Psi}^\dagger+\widetilde{\Psi}\right) \;+\; 2\lambda n\,\widetilde{\Psi}^\dagger\widetilde{\Psi} +\; \tfrac{1}{2}\lambda n\left(\widetilde{\Psi}\widetilde{\Psi}+\widetilde{\Psi}^\dagger\widetilde{\Psi}^\dagger\right)\\ &\quad+\; \lambda\sqrt{n}\left(\widetilde{\Psi}^\dagger\widetilde{\Psi}^\dagger\widetilde{\Psi}+\widetilde{\Psi}^\dagger\widetilde{\Psi}\widetilde{\Psi}\right) \;+\; \tfrac{1}{2}\lambda\,\widetilde{\Psi}^\dagger\widetilde{\Psi}^\dagger\widetilde{\Psi}\widetilde{\Psi}\end{aligned} \tag{S.34}$$

and hence for $\mu = +\lambda n$,

$$\hat{H} = \int d^3\mathbf{x} \left\{ \frac{1}{2M} \nabla\Psi^\dagger \cdot \nabla\Psi \;-\; \mu\Psi^\dagger\Psi \;+\; \tfrac{1}{2}\lambda\Psi^\dagger\Psi^\dagger\Psi\Psi \right\}$$
$$= E_0 \;+\; hatH_2 \;+\; hatH_{\rm int} \tag{S.35}$$

where $E_0 = -\frac{1}{2}\lambda n^2 \times \text{volume}$, $\hat{H}_2$ is exactly as in eq. (17), and

$$\hat{H}_{\rm int} = \int d^3\mathbf{x} \left\{ \lambda\sqrt{n} \left(\widetilde{\Psi}^\dagger\widetilde{\Psi}^\dagger\widetilde{\Psi} + \widetilde{\Psi}^\dagger\widetilde{\Psi}\widetilde{\Psi} \right) \;+\; \tfrac{1}{2}\lambda\, \widetilde{\Psi}^\dagger\widetilde{\Psi}^\dagger\widetilde{\Psi}\widetilde{\Psi} \right\} \tag{S.36}$$

comprises interaction terms for the shifted fields. Physically, $\hat{H}_2$ gives rise to quasiparticle excitations in the superfluid while $\hat{H}_{\rm int}$ is responsible for the interactions between such quasiparticles.

Problem **3**(b):

Obviously, the shifted operators $\tilde{a}_{\mathbf{k}}$ and $\tilde{a}^\dagger_{\mathbf{k}}$ satisfy the same commutation relations as the un-shifted annihilation and creation operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$, so the real issue in this problem is the canonical transform (18).

Generally, a canonical transformation is a linear, simplectic transformation of the coordinates and momenta (of some mechanical system) into each other. In terms of creation and annihilation operators, a generic canonical transform amounts to

$$\hat{b}_i = \sum_j \bigl(u_{ij}\hat{a}_j + v_{ij}\hat{a}^\dagger_j \bigr), \qquad \hat{b}^\dagger_i = \sum_j \bigl(u^*_{ij}\hat{a}^\dagger_j + v^*_{ij}\hat{a}_j \bigr) \tag{S.37}$$

where the coefficient matrices u and v are chosen such that the $\hat{b}_i$ and $\hat{b}^\dagger_i$ operators satisfy the same bosonic commutation relations as the original $\hat{a}_i$ and $\hat{a}^\dagger_i$ operators. In matrix language, we need

$$uu^\dagger \;-\; vv^\dagger \;=\; 1, \qquad uv^\top \;-\; vu^\top \;=\; 0. \tag{S.38}$$

The first of these conditions provides for

$$[\hat{b}_i, \hat{b}^\dagger_j] = \sum_k \bigl(u_{ik}u^*_{jk} \;-\; v_{ik}v^*_{jk} \bigr) \;\equiv\; \bigl(uu^\dagger \;-\; vv^\dagger \bigr)_{ij} \;=\; \delta_{ij}$$

while the second condition provides for

$$[\hat{b}_i, \hat{b}_j] \;=\sum_k \bigl(u_{ik}v_{jk} \;-\; v_{ik}u_{jk}\bigr) \;\equiv\; \bigl(uv^\top \;-\; vu^\top\bigr)_{ij} \;=\; 0$$

(and hence via hermitian conjugation, $[\hat{b}_i^\dagger, \hat{b}_j^\dagger] = 0$).

For the problem at hand, the transform (18) is a special case of (S.37) with block-diagonal u and v matrices where each block comprises just two modes, namely $+\mathbf{k}$ and $-\mathbf{k}$. Given $t_{-\mathbf{k}} = t_{+\mathbf{k}}$, each block looks like

$$u \;=\; \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \qquad v \;=\; \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} \tag{S.39}$$

where $c = \cosh(t_{\mathbf{k}})$ and $s = \sinh(t_{\mathbf{k}})$. The blocks (S.39) clearly satisfy both conditions (S.38) (note $c^2 - s^2 = 1$), hence the entire matrices u and v satisfy (S.38) as well and the operators $\hat{b}_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}^\dagger$ satisfy the bosonic commutation relations.

Problem **3**(c):

According to eqs. (18),

$$\begin{aligned} \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} \;=\;& \cosh^2(t_{\mathbf{k}})\,\tilde{a}_{\mathbf{k}}^\dagger \tilde{a}_{\mathbf{k}} \;+\; \sinh^2(t_{\mathbf{k}})\,\bigl(\tilde{a}_{-\mathbf{k}}\tilde{a}_{-\mathbf{k}}^\dagger \;=\; \tilde{a}_{-\mathbf{k}}^\dagger \tilde{a}_{-\mathbf{k}} \,+\, V\bigr) \\ &+\; \cosh(t_{\mathbf{k}})\sinh(t_{\mathbf{k}})\,\bigl(\tilde{a}_{-\mathbf{k}}\tilde{a}_{\mathbf{k}} \;+\; \tilde{a}_{\mathbf{k}}^\dagger \tilde{a}_{-\mathbf{k}}^\dagger\bigr) \end{aligned} \tag{S.40}$$

and therefore

$$\begin{aligned} \bigl(\hat{b}_{+\mathbf{k}}^\dagger \hat{b}_{+\mathbf{k}} \;+\; \hat{b}_{-\mathbf{k}}^\dagger \hat{b}_{-\mathbf{k}}\bigr) \;=\;& \cosh(2t_{\mathbf{k}})\,\bigl(\tilde{a}_{+\mathbf{k}}^\dagger \tilde{a}_{+\mathbf{k}} \;+\; \tilde{a}_{-\mathbf{k}}^\dagger \tilde{a}_{-\mathbf{k}}\bigr) \\ &+\; \sinh(2t_{\mathbf{k}})\,\bigl(\tilde{a}_{-\mathbf{k}}\tilde{a}_{+\mathbf{k}} \;+\; \tilde{a}_{+\mathbf{k}}^\dagger \tilde{a}_{-\mathbf{k}}^\dagger\bigr) \\ &+\; 2V\sinh^2(t_{\mathbf{k}}) \end{aligned} \tag{S.41}$$

where $V \;=\; [\tilde{a}_{\mathbf{k}}, \tilde{a}_{\mathbf{k}}^\dagger] \;=\; (2\pi)^3\delta^{(3)}(\mathbf{0}) \;=$ volume of the superfluid. Consequently, as long as $t_{-\mathbf{k}} = t_{+\mathbf{k}}$ and $\omega_{-\mathbf{k}} = \omega_{+\mathbf{k}}$,

$$\begin{aligned} &\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\Bigl(\omega_{\mathbf{k}}\cosh(2t_k)\,\tilde{a}_{\mathbf{k}}^\dagger \tilde{a}_{\mathbf{k}} \;+\; \tfrac{1}{2}\omega_{\mathbf{k}}\sinh(2t_{\mathbf{k}})\,\bigl(\tilde{a}_{-\mathbf{k}}\tilde{a}_{+\mathbf{k}} + \tilde{a}_{+\mathbf{k}}^\dagger \tilde{a}_{-\mathbf{k}}^\dagger\bigr)\Bigr) \\ &\qquad\qquad =\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\omega_k\,\hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} \;+\; \text{const.} \end{aligned} \tag{S.42}$$

Let us now take another look at the 'free' Hamiltonian (17) and expand it in terms of the

$\tilde{a}_{\mathbf{k}}$ and $\tilde{a}^{\dagger}_{\mathbf{k}}$ modes of the shifted fields:

$$\hat{H}_2 \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\left(\left(\frac{\mathbf{k}^2}{2M}+\lambda n\right)\tilde{a}^{\dagger}_{\mathbf{k}}\tilde{a}_{\mathbf{k}} \;+\; \tfrac{1}{2}\lambda n\left(\tilde{a}_{-\mathbf{k}}\tilde{a}_{+\mathbf{k}}+\tilde{a}^{\dagger}_{+\mathbf{k}}\tilde{a}^{\dagger}_{-\mathbf{k}}\right)\right), \tag{S.43}$$

and therefore, in light of eq. (S.42),

$$\hat{H}_2 \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\omega_{\mathbf{k}}\hat{b}^{\dagger}_{\mathbf{k}}\hat{b}_{\mathbf{k}} \;+\; \text{const} \tag{19}$$

where

$$\omega_{\mathbf{k}}\cosh(2t_k) \;=\; \frac{\mathbf{k}^2}{2M}+\lambda n, \qquad \omega_{\mathbf{k}}\sinh(2t_k) \;=\; \lambda n. \tag{S.44}$$

Solving these equations for $\omega_{\mathbf{k}}$ and $t_{\mathbf{k}}$ gives us

$$\omega^2_{\mathbf{k}} \;=\; \left(\frac{\mathbf{k}^2}{2M}+\lambda n\right)^2 \;-\; (\lambda n)^2 \;=\; \frac{\mathbf{k}^2}{2M}\left(\frac{\mathbf{k}^2}{2M}+2\lambda n\right), \tag{S.45}$$

cf. eq. (20), and

$$\tanh(2t_k) \;=\; \frac{2M\lambda n}{\mathbf{k}^2+2M\lambda n} \quad i.e., \quad t_{\mathbf{k}} \;=\; \tfrac{1}{4}\log\frac{\mathbf{k}^2+4M\lambda n}{\mathbf{k}^2}\,. \tag{S.46}$$

Aside: Note that at this level of analysis (*i.e.*, ignoring the quasiparticle interactions following from the $\hat{H}_{\text{int}}$), the frequencies (20) are simply the frequencies of the plane-wave solutions to the linearized classical field equations for the shifted fields. Indeed, expanding the classical Landau–Ginzburg field equations

$$\begin{aligned} i\frac{\partial}{\partial t}\Phi(\mathbf{x},t) \;&=\; -\frac{\nabla^2}{2M}\Phi \;+\; \lambda(\Phi^*\Phi-n)\Phi,\\ -i\frac{\partial}{\partial t}\Phi^*(\mathbf{x},t) \;&=\; -\frac{\nabla^2}{2M}\Phi^* \;+\; \lambda(\Phi^*\Phi-n)\Phi, \end{aligned} \tag{S.47}$$

to first order in $\widetilde{\Phi}(\mathbf{x},t)\Phi(\mathbf{x},t)-\sqrt{n}$, gives us

$$\begin{aligned} i\frac{\partial}{\partial t}\widetilde{\Phi}(\mathbf{x},t) \;&=\; -\frac{\nabla^2}{2M}\widetilde{\Phi} \;+\; \lambda n\big(\widetilde{\Phi}^*+\widetilde{\Phi}\big),\\ -i\frac{\partial}{\partial t}\widetilde{\Phi}^*(\mathbf{x},t) \;&=\; -\frac{\nabla^2}{2M}\widetilde{\Phi}^* \;+\; \lambda n\big(\widetilde{\Phi}^*+\widetilde{\Phi}\big), \end{aligned} \tag{S.48}$$

and the plane-wave solutions of these equations look like

$$\begin{aligned}\widetilde{\Phi}(\mathbf{x},t) &= A\cosh(t)\,e^{+i\mathbf{kx}-i\omega t} \;+\; A^*\sinh(t)\,e^{-i\mathbf{kx}+i\omega t},\\ \widetilde{\Phi}^*(\mathbf{x},t) &= A\sinh(t)\,e^{+i\mathbf{kx}-i\omega t} \;+\; A^*\cosh(t)\,e^{-i\mathbf{kx}+i\omega t},\end{aligned}\tag{S.49}$$

where ω, t and $\mathbf{k}$ are related to each other exactly as in eqs. (20) and (S.46).

Furthermore, in the free-shifted-fields approximation (*i.e.*, ignoring the $\hat{H}_{\text{int}}$), the time-dependent quantum fields expands into annihilation and creation operators according to

$$\begin{aligned}\widetilde{\Psi}(\mathbf{x},t) &= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\left(\cosh(t)\,e^{+i\mathbf{kx}-i\omega t}\,\hat{b}_{\mathbf{k}}(0) \;+\; \sinh(t)\,e^{-i\mathbf{kx}+i\omega t}\,\hat{b}^\dagger_{\mathbf{k}}(0)\right),\\ \widetilde{\Psi}^\dagger(\mathbf{x},t) &= \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\left(\sinh(t)\,e^{+i\mathbf{kx}-i\omega t}\,\hat{b}_{\mathbf{k}}(0) \;+\; \cosh(t)\,e^{-i\mathbf{kx}+i\omega t}\,\hat{b}^\dagger_{\mathbf{k}}(0)\right),\end{aligned}\tag{S.50}$$

where $\omega = \omega_{\mathbf{k}}$ and $t = t_{\mathbf{k}}$, exactly as in classical solutions (S.49). Indeed, the free quantum fields (S.50) are precisely linear combinations of the free classical plane-wave solutions with operatorial coefficients — exactly as the free relativistic quantum fields we have studied in class and the relativistic vector fields discussed in problem 1 of this homework, cf eq. (12). Moreover, for both relativistic and non-relativistic quantum fields, the positive-frequency solutions ($e^{-i\omega t}$ with $\omega > 0$) are accompanied by the annihilation operators ($\hat{a}_{\mathbf{k},\lambda}$ for the vector fields or $\hat{b}_{\mathbf{k}}$ for Helium) while the negative-frequency solutions ($e^{+i\omega t}$) are accompanied by the creation operators. These rules are universal in Quantum Field Theory, relativistic or otherwise. The only difference is the extra factor $1/\sqrt{2\omega_{\mathbf{k},\lambda}}$ in the relativistic theories.

Optional problem **3**(*):

First, note that the operator $\hat{F}$ is anti-hermitian, $\hat{F}^\dagger = -\hat{F}$, hence $e^{\hat{F}}$ is a unitary operator, which means that the operator transform

$$\tilde{a}_{\mathbf{k}} \;\to\; \hat{b}_{\mathbf{k}} \;=\; e^{\hat{F}}\tilde{a}_{\mathbf{k}}e^{-\hat{F}}, \qquad \tilde{a}^\dagger_{\mathbf{k}} \;\to\; \hat{b}^\dagger_{\mathbf{k}} \;=\; e^{\hat{F}}\tilde{a}^\dagger_{\mathbf{k}}e^{-\hat{F}}\tag{S.51}$$

is indeed consistent with the hermitian conjugation. The operators $\hat{b}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$ *defined* by eqs. (S.51) obviously satisfy the same bosonic commutation relations as the $\tilde{a}_{\mathbf{k}}$ and $\tilde{a}^\dagger_{\mathbf{k}}$ operators, and just as obviously, the operators $\hat{b}_{\mathbf{k}}$ annihilate the ground state (quasi-particle vacuum) $|\Omega_2\rangle = e^{\hat{F}}\,|\text{coh}\rangle$ simply because the $\tilde{a}_{\mathbf{k}}$ operators annihilate the coherent state $|\text{coh}\rangle$. What is not so obvious is that eqs. (S.51) for the $\hat{b}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$ operators agree with eqs. (18).

To verify the agreement, we use the multiple-commutator formula

$$e^{\hat{F}}\hat{B}e^{-\hat{F}} = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\hat{F},\left[\hat{F},\left[\ldots,\left[\hat{F},\hat{B}\right]\ldots\right]\right]\right]_{n\,\text{times}}. \tag{S.52}$$

It is easy to calculate the single commutators

$$[\hat{F},\tilde{a}_{\mathbf{k}}] = t_{\mathbf{k}}\tilde{a}^\dagger_{-\mathbf{k}} \quad\text{and}\quad [\hat{F},\tilde{a}^\dagger_{\mathbf{k}}] = t_{\mathbf{k}}\tilde{a}_{-\mathbf{k}}, \tag{S.53}$$

which give us the multiple commutators

$$\left[\hat{F},\left[\hat{F},\left[\ldots,\left[\hat{F},\tilde{a}_{\mathbf{k}}\right]\ldots\right]\right]\right]_{n\,\text{times}} = \begin{cases} t^n\tilde{a}_{\mathbf{k}} & \text{for even } n,\\ t^n\tilde{a}^\dagger_{-\mathbf{k}} & \text{for odd } n,\end{cases} \tag{S.54}$$

and consequently (according to eq. (S.52))

$$e^{\hat{F}}\tilde{a}_{\mathbf{k}}e^{-\hat{F}} = \sum_{\text{even}\,n}\frac{t^n}{n!}\tilde{a}_{\mathbf{k}} + \sum_{\text{odd}\,n}\frac{t^n}{n!}\tilde{a}^\dagger_{-\mathbf{k}} = \cosh(t_k)\,\tilde{a}_{\mathbf{k}} + \sinh(t_{\mathbf{k}})\,\tilde{a}^\dagger_{-\mathbf{k}}. \tag{S.55}$$

Likewise,

$$e^{\hat{F}}\tilde{a}^\dagger_{\mathbf{k}}e^{-\hat{F}} = \sum_{\text{even}\,n}\frac{t^n}{n!}\tilde{a}^\dagger_{\mathbf{k}} + \sum_{\text{odd}\,n}\frac{t^n}{n!}\tilde{a}_{-\mathbf{k}} = \cosh(t_k)\,\tilde{a}^\dagger_{\mathbf{k}} + \sinh(t_{\mathbf{k}})\,\tilde{a}_{-\mathbf{k}}, \tag{S.56}$$

in full agreement with eqs. (18). $\mathcal{Q.E.D.}$

Problem **3**(d):

Taking a difference between eqs. (S.40) for the $+\mathbf{k}$ and $-\mathbf{k}$ modes we immediately obtain

$$\left(\hat{b}^\dagger_{+\mathbf{k}}\hat{b}_{+\mathbf{k}} + \hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}}\right) = \left(\tilde{a}^\dagger_{+\mathbf{k}}\tilde{a}_{+\mathbf{k}} + \tilde{a}^\dagger_{-\mathbf{k}}\tilde{a}_{-\mathbf{k}}\right). \tag{S.57}$$

Consequently,

$$\begin{aligned}\hat{\mathbf{P}}_{\text{mech}} &= \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\mathbf{k}\,\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\mathbf{k}\,\tilde{a}^\dagger_{\mathbf{k}}\tilde{a}_{\mathbf{k}}\\ &= \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{\mathbf{k}}{2}\left(\tilde{a}^\dagger_{+\mathbf{k}}\tilde{a}_{+\mathbf{k}} - \tilde{a}^\dagger_{-\mathbf{k}}\tilde{a}_{-\mathbf{k}}\right)\\ &= \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{\mathbf{k}}{2}\left(\hat{b}^\dagger_{+\mathbf{k}}\hat{b}_{+\mathbf{k}} - \hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}}\right)\\ &= \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\mathbf{k}\,\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}.\end{aligned} \tag{S.58}$$

$\mathcal{Q.E.D.}$

Problem $\mathbf{3}$(e):

Non-relativistic mechanics — classical or quantum — is invariant under Galilean boosts, which act on coordinates, momenta and energy according to

$$t' = t, \quad \mathbf{x}'_i = \mathbf{x}_i + \mathbf{v}t, \quad \mathbf{p}'_i = \mathbf{p}_i + m_i\mathbf{v}, \quad H' = H + \mathbf{v}\cdot\mathbf{P}_{\text{total}} + \text{const.} \tag{S.59}$$

A superfluid flowing at a uniform velocity $\mathbf{v}$ is related to the same superfluid at rest via a Galilean boost. Consequently, the lab-frame Hamiltonian of the flowing superfluid is

$$\begin{aligned} \big(\hat{H} - \mu\hat{N}\big)^{\text{lab}}_{\text{frame}} &= \big(\hat{H} - \mu\hat{N}\big)^{\text{fluid}}_{\text{frame}} + \mathbf{v}\cdot\hat{\mathbf{P}}^{\text{fluid}}_{\text{frame}} + \text{const} \\ &= \big(\hat{H}_2 + \mathbf{v}\cdot\hat{\mathbf{P}}\big)^{\text{fluid}}_{\text{frame}} + \hat{H}_{\text{int}} + \text{const} \end{aligned} \tag{S.60}$$

where

$$\hat{H}_2^{\text{lab frame}} = \big(\hat{H}_2 + \mathbf{v}\cdot\hat{\mathbf{P}}\big)^{\text{fluid}}_{\text{frame}} = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,(\omega_{\mathbf{k}} + \mathbf{v}\cdot\mathbf{k})\,\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\,. \tag{S.61}$$

(The quasiparticle momenta $\mathbf{k}$ in this formula are in the fluid's rest frame.)

Eq. (S.61) applies to a flow of an ideal gas just as well as a flow of a superfluid, the only difference being in the dispersion relations $\omega(k)$. For the ideal gas, $\omega = k^2/2M$, and therefore for any non-zero velocity $\mathbf{v}$, there are modes $\mathbf{k}$ for which

$$\omega_{\mathbf{k}} + \mathbf{v}\cdot\mathbf{k} = \frac{(\mathbf{k} - M\mathbf{v})^2}{2M} - \tfrac{1}{2}M\mathbf{v}^2 < 0. \tag{S.62}$$

For such modes, excitations created by the $\hat{b}^\dagger_{\mathbf{k}}$ have negative energy, which means that any perturbation of the flowing gas can create such negative-energy excitations and slow down the flow.

For the superfluid however, $\omega_{\mathbf{k}} \geq v_0|\mathbf{k}|$ for all modes, hence

$$\omega_{\mathbf{k}} + \mathbf{v}\cdot\mathbf{k} \geq (v_0 - |\mathbf{v}|)|\mathbf{k}| \geq 0 \tag{S.63}$$

for all the excitation modes, provided $|\mathbf{v}| \leq v_0$. Consequently, *all* excitations in the flowing superfluid have positive lab-frame energies, so they don't get spontaneously created and the flow persists forever without dissipation, thus *superfluidity.*