Problem **1**(a):

Let $\Delta T^{\mu\nu} = \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu}$. Regardless of the specific form of the $\mathcal{K}^{[\lambda\mu]\nu}(\phi,\partial\phi)$ tensor, its anti-symmetry with respect to its first two indices implies

$$\partial_\mu \Delta T^{\mu\nu} \;=\; \partial_\mu\partial_\lambda \mathcal{K}^{[\lambda\mu]\nu} \;=\; 0 \tag{S.1}$$

and hence the first eq. (3). Furthermore,

$$\int d^3\mathbf{x}\, \left(\Delta T^{0\nu} \;=\; \partial_i \mathcal{K}^{i0\nu}\right) \;=\; \oint\limits_{\substack{\text{boundary}\\ \text{of space}}} d^2\text{Area}_i\, \mathcal{K}^{i0\nu} \;\longrightarrow\; 0 \tag{S.2}$$

when the integral is taken over the whole space, hence the second eq. (3).

Problem **1**(b):

In the Noether's formula (1) for the stress-energy tensor, ϕ_a stand for the independent fields, however labelled. In the electromagnetic case, the independent fields are components of the 4–vector $A_\lambda(x)$, hence

$$\begin{aligned} T^{\mu\nu}_{\text{Noether}}(\text{EM}) \;&=\; \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\,\partial^\nu A_\lambda \;-\; g^{\mu\nu}\,\mathcal{L} \\ &=\; -F^{\mu\lambda}\,\partial^\nu A_\lambda \;+\; \tfrac{1}{4} g^{\mu\nu}\,F_{\kappa\lambda}F^{\kappa\lambda}. \end{aligned} \tag{S.3}$$

While the second term here is clearly both gauge invariant and symmetric in $\mu \leftrightarrow \nu$, the first term is neither.

Problem **1**(c):

Clearly, one can easily restore both symmetry and gauge invariance of the electromagnetic stress-energy tensor by replacing $\partial^\nu A_\lambda$ in eq. (S.3) with $F^{\nu}{}_{\lambda}$, hence eq. (5). The correction amounts to

$$T^{\mu\nu} - T^{\mu\nu}_{\text{Noether}} \;=\; -F^{\mu\lambda}\left(F^{\nu}{}_{\lambda} - \partial^\nu A_\lambda \;=\; -\partial_\lambda A^\nu\right)\big) \;=\; \partial_\lambda\left(F^{\mu\lambda}A^\nu \;\stackrel{\text{def}}{=}\; \mathcal{K}^{[\lambda\mu]\nu}\right) - A^\nu\left(\partial_\lambda F^{\mu\lambda}\right). \tag{S.4}$$

Since the free electromagnetic field satisfies $\partial_\lambda F^{\mu\lambda} = 0$, the second term on the right hand side here vanishes — and the remaining correction indeed has form (2).

Problem **1**(d):

As explained in class, the Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\kappa\lambda}F^{\kappa\lambda} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)$. Combining this fact with eq. (5), we have the energy density

$$\mathcal{H} \;=\; T^{00} \;=\; -F^{0i}F^{0}_{\;i} \;-\; \mathcal{L} \;=\; +\mathbf{E}^2 \;-\; \tfrac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) \;=\; \tfrac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) \tag{S.5}$$

in agreement with the standard electromagnetic formulæ (note the $c = 1$, *rationalized* units here). Likewise, the energy flux and the momentum density are

$$S^i \;=\; T^{i0} \;=\; T^{0i} \;=\; -F^{0j}F^{i}_{\;j} \;=\; -(-E^j)(\epsilon^{ijk}B^k) \;=\; +\epsilon^{ijk}E^jB^k \;=\; (\mathbf{E}\times\mathbf{B})^i, \tag{S.6}$$

in agreement with the Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{B}$ (again, in the $c = 1$, *rationalized* units). Finally, the (3–dimensional) stress tensor is

$$\begin{aligned} T^{ij}_{\rm EM} \;&=\; -F^{i\lambda}F^{j}_{\;\lambda} \;-\; g^{ij}\mathcal{L} \;=\; -F^{i0}F^{j}_{\;0} \;-\; F^{ik}F^{j}_{\;k} \;+\; \delta^{ij}\mathcal{L} \\ &=\; -E^iE^j \;+\; \epsilon^{ik\ell}B^\ell\,\epsilon^{jkm}B^m \;+\; \tfrac{1}{2}\delta^{ij}\left(\mathbf{E}^2 - \mathbf{B}^2\right) \\ &=\; -E^iE^j \;-\; B^iB^j \;+\; \tfrac{1}{2}\delta^{ij}\left(\mathbf{E}^2 + \mathbf{B}^2\right). \end{aligned} \tag{S.7}$$

Problem **2**(a):

In a sense, eq. (6) follows from eq. (S.4), but it is just as easy to derive it directly from Maxwell equations. Starting with eq. (5), we immediately have

$$\partial_\mu T^{\mu\nu}_{\rm EM} \;=\; -\big(\partial_\mu F^{\mu\lambda}\big)F^{\nu}_{\;\lambda} \;-\; F^{\mu\lambda}\big(\partial_\mu F^{\nu}_{\;\lambda}\big) \;+\; \tfrac{1}{2}F_{\kappa\lambda}\big(\partial^\nu F^{\kappa\lambda}\big). \tag{S.8}$$

Using the antisymmetry $F^{\mu\lambda} = -F^{\lambda\mu}$, the second term on the right hand side here becomes

$$-\,F^{\mu\lambda}\,\partial_\mu F^{\nu}_{\;\lambda} \;=\; +F_{\mu\lambda}\,\partial^\mu F^{\lambda\nu} \;=\; +F_{\mu\lambda}\,\partial^\lambda F^{\nu\mu} \;=\; \tfrac{1}{2}F_{\mu\lambda}\big(\partial^\lambda F^{\nu\mu} + \partial^\mu F^{\lambda\nu}\big) \;=\; -\tfrac{1}{2}F_{\mu\lambda}\big(\partial^\nu F^{\mu\lambda}\big) \tag{S.9}$$

(using Maxwell equation $\partial^\lambda F^{\nu\mu} + \partial^\mu F^{\lambda\nu} + \partial^\nu F^{\mu\lambda} = 0$) and thus precisely cancels the third term on the right hand side of eq. (S.8). Consequently,

$$\partial_\mu T^{\mu\nu}_{\rm EM} \;=\; -\big(\partial_\mu F^{\mu\lambda}\big)F^{\nu}_{\;\lambda} \;=\; -\;J^\lambda\,F^{\nu}_{\;\lambda} \tag{S.10}$$

by the other Maxwell equation $\partial_\mu F^{\mu\lambda} = J^\lambda$. $\quad\mathcal{Q.E.D.}$

Problem **2**(b):

From the spacetime point of view, a covariant derivative acting upon a field Φ_q of charge q is a differential operator $D_\mu = \partial_\mu + iqA_\mu(x)$. Consequently,

$$\begin{aligned} [D_\mu, D_\nu] &= [\partial_\mu, \partial_\nu] + iq[\partial_\mu, A_\nu(x)] + iq[A_\mu(x), \partial_\nu] - q^2[A_\mu(x), A_\nu(x)] \\ &= 0 + iq\,(\partial_\mu A_\nu(x)) + iq\,(-\partial_\nu A_\mu(x)) - q^2(0) \\ &= iq\,(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)) = iq\,F_{\mu\nu}(x). \end{aligned} \tag{S.11}$$

$\mathcal{Q.E.D.}$

Problem **2**(c):

According to the Noether theorem (1),

$$\begin{aligned} T^{\mu\nu}_{\text{Noether}} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\,\partial^\nu A_\lambda + \frac{\partial\mathcal{L}}{\partial(\partial_\mu \Phi)}\,\partial^\nu\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu \Phi^*)}\,\partial^\nu\Phi^* - g^{\mu\nu}\,\mathcal{L} \\ &= T^{\mu\nu}_{\text{Noether}}(\text{EM}) + T^{\mu\nu}_{\text{Noether}}(\text{mat}) \end{aligned} \tag{S.12}$$

where $T^{\mu\nu}_{\text{Noether}}(\text{EM})$ is exactly as in eq. (S.3) for free EM fields and

$$T^{\mu\nu}_{\text{Noether}}(\text{mat}) = D^\mu\Phi^*\,\partial^\nu\Phi + D^\mu\Phi\,\partial^\nu\Phi^* - g^{\mu\nu}\big(D^\lambda\Phi^*\,D_\lambda\Phi - m^2\Phi^*\Phi\big). \tag{S.13}$$

Both terms on the second line of eq. (S.12) lack $\mu \leftrightarrow \nu$ symmetry and gauge invariance and thus need corrections à la (2). In fact, the same $\mathcal{K}^{[\lambda\mu]\nu} = F^{\mu\lambda}A^\nu$ we used to improve the free electromagnetic stress-energy tensor will now improve both the $T^{\mu\nu}_{\text{EM}}$ and $T^{\mu\nu}_{\text{mat}}$ at the same time! Indeed, according to eq. (S.4),

$$\partial_\lambda\left(F^{\mu\lambda}A^\nu \stackrel{\text{def}}{=} \mathcal{K}^{[\lambda\mu]\nu}\right) = T^{\mu\nu}(\text{EM}) - T^{\mu\nu}_{\text{Noether}}(\text{EM}) + A^\nu\left(\partial_\lambda F^{\mu\lambda}\right), \tag{S.14}$$

where the last term on the right hand side is precisely the difference between the improved stress-energy tensor (11) for the charged fields and the Noether tensor (S.13) for the same.

The proof of the last assertion is based upon Maxwell equation

$$\begin{aligned}\partial_\lambda F^{\mu\lambda} &= -J^\mu = +\frac{\partial\mathcal{L}}{\partial A_\mu} \\ &= \frac{\partial D_\nu\Phi^*}{\partial A_\mu} D^\nu\Phi + \frac{\partial D_\nu\Phi}{\partial A_\mu} D^\nu\Phi^* \\ &= -iq\Phi^* D^\mu\Phi + iq\Phi D^\nu\Phi^*.\end{aligned} \tag{S.15}$$

Consequently,

$$\begin{aligned}A^\nu\left(\partial_\lambda F^{\lambda\mu}\right) &= \bigl(-iqA^\nu\Phi^*\bigr)D^\mu\Phi + \bigl(+iqA^\nu\Phi\bigr)D^\mu\Phi^* \\ &= D^\mu\Phi\bigl(D^\nu\Phi^* - \partial^\nu\Phi^*\bigr) + D^\mu\Phi^*\bigl(D^\nu\Phi - \partial^\nu\Phi\bigr) \\ &= T^{\mu\nu}(\text{mat}) - T^{\mu\nu}_{\text{Noether}}(\text{mat}).\end{aligned} \tag{S.16}$$

$\mathcal{Q.E.D.}$

Problem **2**(d):

Because the fields $\Phi(x)$ and $\Phi^*(x)$ have opposite electric charges, their product is neutral and therefore $\partial_\mu(\Phi^*\Phi) = D_\mu(\Phi^*\Phi) = (D_\mu\Phi^*)\Phi + \Phi^*(D_\mu\Phi)$. Similarly,

$$\begin{aligned}\partial_\mu\left((D^\mu\Phi^*)\,(D^\nu\Phi)\right) &= (D_\mu D^\mu\Phi^*)\,(D^\nu\Phi) + (D^\mu\Phi^*)\,(D_\mu D^\nu\Phi) \\ &= -m^2\Phi^*\,(D^\nu\Phi) + (D_\mu\Phi^*)\,(D^\nu D^\mu\Phi + iqF^{\mu\nu}\Phi)\end{aligned} \tag{S.17}$$

where we have applied the field equation $\bigl(D_\mu D^\mu + m^2\bigr)\Phi^*(x) = 0$ to the first term on the right hand side and used eq. (8) to expand the second term. Likewise,

$$\begin{aligned}\partial_\mu\left((D^\mu\Phi)\,(D^\nu\Phi^*)\right) &= (D_\mu D^\mu\Phi)\,(D^\nu\Phi^*) + (D^\mu\Phi)\,(D_\mu D^\nu\Phi^*) \\ &= -m^2\Phi\,(D^\nu\Phi^*) + (D_\mu\Phi)\,(D^\nu D^\mu\Phi^* - iqF^{\mu\nu}\Phi^*),\end{aligned} \tag{S.18}$$

and finally

$$\begin{aligned}\partial_\mu\left[-g^{\mu\nu}\left(D_\lambda\Phi^* D^\lambda\Phi - m^2\,\Phi^*\Phi\right)\right] &= -\partial^\nu\left(D_\lambda\Phi^* D^\lambda\Phi\right) + m^2\partial^\nu(\Phi^*\Phi) \\ &= -(D^\nu D^\mu\Phi^*)\,(D_\mu\Phi) - (D_\mu\Phi^*)\,(D^\nu D^\mu\Phi) \\ &\quad + m^2\Phi\,(D^\nu\Phi^*) + m^2\Phi^*\,(D^\nu\Phi).\end{aligned} \tag{S.19}$$

Together, the left hand sides of eqs. (S.17), (S.18) and (S.19) comprise $\partial_\mu T^{\mu\nu}_{\text{mat}}$ — *cf.* eq. (11). On the other hand, combining the right hand sides of these three equations results in massive cancellation of all terms except those containing the gauge field strength tensor $F^{\mu\nu}$. Hence,

$$\partial_\mu T^{\mu\nu}_{\rm mat} \;=\; iqF^{\mu\nu}\left(\Phi\, D_\mu\Phi^* \;-\; \Phi^*\, D_\mu\Phi\right) \;=\; F^{\mu\nu}\, J_\nu\,. \tag{S.20}$$

$\mathcal{Q.E.D.}$

Problem $\mathbf{3}$(a):

In terms of momentum-and-polarization modes $E_{\mathbf{k},\lambda}$ and $B_{\mathbf{k},\lambda}$ of the electric and magnetic fields,

$$L \;=\; \int\! d^3\mathbf{x}\left(\mathcal{L}=\tfrac12\mathbf{E}^2-\tfrac12\mathbf{B}^2\right) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{\lambda=-1,0,+1}\left(\tfrac12 E^*_{\mathbf{k},\lambda}E_{\mathbf{k},\lambda} \;-\; \tfrac12 B^*_{\mathbf{k},\lambda}B_{\mathbf{k},\lambda}\right). \tag{S.21}$$

As explained in the solutions to the previous homework,

$$B_{\mathbf{k},\lambda} \;=\; ik\lambda A_{\mathbf{k},\lambda} \quad\text{and}\quad E_{\mathbf{k},\lambda} \;=\; -\dot{A}_{\mathbf{k},\lambda} \;-\; ik\delta_{\lambda,0}A^0_{\mathbf{k}}\,. \tag{S.22}$$

Substituting these values into eq. (S.21) and separating the transverse modes $\lambda=\pm1$ from the longitudinal modes $\lambda=0$, we arrive at eqs. (12–14), $\mathcal{Q.E.D.}$

Problem $\mathbf{3}$(b):

In 3D notations, gauge transforms $A_\mu(x)\to A_\mu(x)-\partial_\mu\Lambda(x)$ look like

$$\mathbf{A}(\mathbf{x},t) \;\mapsto\; \mathbf{A}(\mathbf{x},t) \;+\; \nabla\Lambda(\mathbf{x},t), \quad A^0(\mathbf{x},t) \;\mapsto A^0(\mathbf{x},t) \;-\; \dot\Lambda(\mathbf{x},t). \tag{S.23}$$

In terms of momentum-and-polarization modes, a generic gauge transform (S.23) for an arbitrary $\Lambda(\mathbf{x},t)$ becomes

$$A_{\mathbf{k},\lambda}(t) \;\mapsto\; A_{\mathbf{k},\lambda}(t) \;+\; ik\delta_{\lambda,0}\Lambda_{\mathbf{k}}(t), \quad A^0_{\mathbf{k}}(t) \;\mapsto\; A^0_{\mathbf{k}}(t) \;-\; \dot\Lambda_{\mathbf{k}}(t) \tag{S.24}$$

for arbitrary $\Lambda_{\mathbf{k}}(t)$, hence gauge invariance of the transverse $A_{\mathbf{k},\lambda}(t)$ ($\lambda=\pm1$) and eqs. (15) for the longitudinal and scalar modes.

Problem $\mathbf{3}$(c):

According to eq. (13), canonically conjugate variables to the transverse $A_{\mathbf{k},\lambda}$ are $\Pi_{\mathbf{k},\lambda} = A^*_{\mathbf{k},\lambda} = -E^*_{\mathbf{k},\lambda}$. (Mind the double-counting due to $A^*_{\mathbf{k},\lambda} = -A_{-\mathbf{k},\lambda}$.) Consequently, upon quantization, we have canonical equal-time commutation relations

$$[\hat{A}_{\mathbf{k},\lambda}, \hat{A}^\dagger_{\mathbf{k}',\lambda'}] \;=\; 0, \quad [\hat{E}_{\mathbf{k},\lambda}, \hat{E}^\dagger_{\mathbf{k}',\lambda'}] \;=\; 0, \quad [\hat{A}_{\mathbf{k},\lambda}, \hat{E}^\dagger_{\mathbf{k}',\lambda'}] \;=\; -i\delta_{\lambda,\lambda'}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'), \tag{S.25}$$

where $\hat{A}^\dagger_{\mathbf{k},\lambda} = -\hat{A}_{-\mathbf{k},\lambda}$ and likewise $\hat{E}^\dagger_{\mathbf{k},\lambda} = -\hat{E}_{-\mathbf{k},\lambda}$.

The Hamiltonian for the transverse modes follows from the Lagrangian (13) in an obvious way; upon quantization, we have

$$\hat{H}^\top \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\lambda=\pm1} \left(\tfrac{1}{2}\hat{E}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda} \;+\; \tfrac{1}{2}\mathbf{k}^2\,\hat{A}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda}\right). \tag{S.26}$$

Problem $\mathbf{3}$(d):

Proceeding exactly as in the previous homework about the massive vector fields, we define annihilation and creation operators

$$\hat{a}_{\mathbf{k},\lambda} \;=\; \frac{\omega_{\mathbf{k}}\hat{A}_{\mathbf{k},\lambda} \;-\; i\hat{E}_{\mathbf{k},\lambda}}{\sqrt{2\omega_{\mathbf{k}}}}\,, \qquad \hat{a}^\dagger_{\mathbf{k},\lambda} \;=\; \frac{\omega_{\mathbf{k}}\hat{A}^\dagger_{\mathbf{k},\lambda} \;+\; i\hat{E}^\dagger_{\mathbf{k},\lambda}}{\sqrt{2\omega_{\mathbf{k}}}}\,, \tag{S.27}$$

where $\omega_{\mathbf{k}} = k \equiv |\mathbf{k}|$, as appropriate for the massless particles. The bosonic commutation relations between these operators and eq. (16) for the Hamiltonian (S.26) follow in exactly the same way as for the massive vector particles; see solutions for the previous homework for details.

Problem $\mathbf{3}$(e):

In the transverse gauge $A_{\mathbf{k},0} \equiv 0$, the longitudinal Lagrangian (14) reduces to

$$L^{\|} \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{\mathbf{k}^2}{2}\,\left|A^0_{\mathbf{k}}\right|^2, \tag{S.28}$$

which contains no time derivatives whatsoever and thus yields a time-independent equation of "motion" for the $A^0_{\mathbf{k}}$ modes, namely $\mathbf{k}^2 A^0_{\mathbf{k}} = 0$.

The electric current $J^\mu(x)$ couples to the EM fields via the Lagrangian term $\mathcal{L} \supset -J^\mu A_\mu = \mathbf{J}\cdot\mathbf{A} - J^0 A^0$; in terms of the momentum-and-polarization modes, this interaction term is

$$L_{\text{int}} = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\left(\sum_\lambda A_{\mathbf{k},\lambda}J^*_{\mathbf{k},\lambda} - A^0_{\mathbf{k},\lambda}(J^0_{\mathbf{k},\lambda})^*\right). \tag{S.29}$$

Quantizing the transverse part of this interaction gibes us the interaction Hamiltonian for the photons

$$\hat{H}_{\text{int}} = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{\lambda=\pm1}\left(-\hat{A}_{\mathbf{k},\lambda}\hat{J}^\dagger_{\mathbf{k},\lambda}\right) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{-1}{\sqrt{2\omega_\mathbf{k}}}\sum_{\lambda=\pm1}\left(\hat{a}_{\mathbf{k},\lambda}\hat{J}^\dagger_{\mathbf{k},\lambda} + \hat{a}^\dagger_{\mathbf{k},\lambda}\hat{J}_{\mathbf{k},\lambda}\right), \tag{S.30}$$

which is responsible for emission and absorption of photons by matter (via the current operators $\hat{J}_{\mathbf{k},\lambda} = \hat{J}^\dagger_{-\mathbf{k},\lambda}$).

The longitudinal part of the interaction (S.29) together with the free longitudinal Lagrangian (14) have particularly simple form in the transverse gauge, namely

$$L^{\|} = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\tfrac{1}{2}\left(\mathbf{k}^2\left|A^0_\mathbf{k}\right|^2 - A^0_\mathbf{k}(J^0_\mathbf{k})^* - (A^0_\mathbf{k})^*J^0_\mathbf{k}\right). \tag{S.31}$$

Again, there are no time derivatives in this Lagrangian, hence time-independent equations of "motion" $\mathbf{k}^2 A^0_\mathbf{k} = J^0_\mathbf{k}$, or in spacetime terms, $\nabla^2 A^0(\mathbf{x},t) = -J^0(\mathbf{x},t)$, *i.e.*, $A^0(\mathbf{x},t)$ is the *instantaneous* Coulomb potential for the charge density $J^0(\mathbf{x}', \text{same } t)$. $\mathcal{Q.E.D.}$

Since the A^0 field is non-dynamical, we may substitute the solution of its time-independent equation of "motion" back into the Lagrangian (S.31). The result is

$$\Delta L_{\text{Coulomb}} = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{-\left|J^0_\mathbf{k}\right|^2}{2\mathbf{k}^2} = \iint d^3\mathbf{x}\,d^3\mathbf{y}\,\frac{-1}{8\pi|\mathbf{x}-\mathbf{y}|}\,J^0(\mathbf{x})\,J^0(\mathbf{y}), \tag{S.32}$$

i.e., (minus) the Coulomb energy due to the electric charge density $J^0(\mathbf{x})$. Physically, it is responsible for the Coulomb forces between charged particles giving rise to the $J^\mu(x)$. In the quantum theory of radiation, (the quantum analogue) of $\Delta H_{\text{Coulomb}} = -\Delta L_{\text{Coulomb}}$ is considered a part of the charged particles' Hamiltonian (*e.g.*, the Coulomb potential for the electrons in an atom) while (S.30) is treated as a perturbation giving rise to emission, absorption and scattering of photons by matter.

Problem **3**(f):

The time-dependent quantum EM field $\hat{A}^\mu(x)$ is completely analogous to a the time-dependent massive vector field studied in the previous homework, except that for the massless EM field we should limit the polarization modes to the transverse modes $\lambda = \pm 1$, hence eq. (18).

Actually, eq. (18) applies only for the EM field subject to the transverse gauge condition. Without a gauge condition, one has

$$\begin{aligned} \hat{A}^\mu(x) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{\sqrt{2|\mathbf{k}|}}\Bigg[& e^{-ikx}\Big(\sum_{\lambda=\pm1} f^\mu(\mathbf{k},\lambda)\,\hat{a}_{\mathbf{k},\lambda}(0)\;+\;k^\mu\hat{\Lambda}_{\mathbf{k}}\Big) \\ & +\;e^{+ikx}\Big(\sum_{\lambda=\pm1} f^{*\mu}(\mathbf{k},\lambda)\,\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\;+\;k^\mu\hat{\Lambda}^\dagger_{\mathbf{k}}\Big)\Bigg]_{k^0=+|\mathbf{k}|} \end{aligned} \tag{S.33}$$

without any restrictions whatsoever for the $\hat{\Lambda}_{\mathbf{k}}$ operators. Various gauge conditions relate $\hat{\Lambda}_{\mathbf{k}}$ to the photonic creation / annihilation operators or to other quantum fields; naturally, such relations differ for different gauge conditions.