Problem **1**(a):

By definition, $\hat{S}^{\mu\nu} \stackrel{\text{def}}{=} \frac{i}{4}[\gamma^\mu,\gamma^\nu] = \frac{i}{2}(\gamma^\mu\gamma\nu - g^{\mu\nu})$. Using this formula, we saw in class that $\left[\gamma^\lambda, S^{\mu\nu}\right] = i\left(g^{\lambda\mu}\gamma^\nu - g^{\lambda\nu}\gamma^\mu\right)$. Consequently, by Leibniz rule,

$$\begin{aligned}
\left[\gamma^\kappa\gamma^\lambda, S^{\mu\nu}\right] &= \gamma^\kappa\left[\gamma^\lambda, S^{\mu\nu}\right] + \left[\gamma^\kappa, S^{\mu\nu}\right]\gamma^\lambda \\
&= \gamma^\kappa\left(ig^{\lambda\mu}\gamma^\nu - ig^{\lambda\nu}\gamma^\mu\right) + \left(ig^{\kappa\mu}\gamma^\nu - ig^{\kappa\nu}\gamma^\mu\right)\gamma^\lambda \\
&= ig^{\lambda\mu}\gamma^\kappa\gamma^\nu - ig^{\kappa\nu}\gamma^\mu\gamma^\lambda - ig^{\lambda\nu}\gamma^\kappa\gamma^\mu + ig^{\kappa\mu}\gamma^\nu\gamma^\lambda \\
&= ig^{\lambda\mu}\left(\gamma^\kappa\gamma^\nu - g^{\kappa\nu}\right) - ig^{\kappa\nu}\left(\gamma^\mu\gamma^\lambda - g^{\lambda\mu}\right) \\
&\qquad - ig^{\lambda\nu}\left(\gamma^\kappa\gamma^\mu - g^{\kappa\mu}\right) + ig^{\kappa\mu}\left(\gamma^\nu\gamma^\lambda - g^{\lambda\nu}\right) \\
&= 2g^{\lambda\mu}S^{\kappa\nu} - 2g^{\kappa\nu}S^{\mu\lambda} - 2g^{\lambda\nu}S^{\kappa\mu} + 2g^{\kappa\mu}S^{\nu\lambda},
\end{aligned}$$

and therefore,

$$\begin{aligned}
\left[S^{\kappa\lambda}, S^{\mu\nu}\right] &= \tfrac{i}{2}\left[\gamma^\kappa\gamma^\lambda, S^{\mu\nu}\right] \\
&= ig^{\lambda\mu}S^{\kappa\nu} - ig^{\kappa\nu}S^{\mu\lambda} - ig^{\lambda\nu}S^{\kappa\mu} + ig^{\kappa\mu}S^{\nu\lambda} \qquad \text{(S.1)} \\
&= ig^{\lambda\mu}S^{\kappa\nu} + ig^{\kappa\nu}S^{\lambda\mu} - ig^{\lambda\nu}S^{\kappa\mu} - ig^{\kappa\mu}S^{\lambda\nu}.
\end{aligned}$$

$\mathcal{Q.E.D.}$

Problem **1**(b): Let $F = -\frac{i}{2}\theta_{\alpha\beta}S^{\alpha\beta}$, thus $M = e^F$ and $M^{-1} = e^{-F}$. We shall use the multiple-commutator formula for the $e^{-F}\gamma^\mu e^{+F}$, so we begin by evaluating the single commutator

$$\left[\gamma^\mu, F\right] = -\tfrac{i}{2}\theta_{\alpha\beta}\left[\gamma^\mu, F\right] = \tfrac{1}{2}\theta_{\alpha\beta}\left(g^{\mu\alpha}\gamma^\beta - g^{\mu\beta}\gamma^\alpha\right) = \tfrac{1}{2}\theta^\mu_{\ \beta}\gamma^\beta - \tfrac{1}{2}\theta_\alpha^{\ \mu}\gamma^\alpha = \theta^\mu_{\ \nu}\gamma^\nu. \qquad \text{(S.2)}$$

The multiple commutators follow immediately from this formula,

$$\begin{aligned}
\left[\left[\gamma^\mu, F\right], F\right] &= \theta^\mu_{\ \lambda}\theta^\lambda_{\ \nu}\gamma^\nu, \\
\left[\left[\left[\gamma^\mu, F\right], F\right], F\right] &= \theta^\mu_{\ \lambda}\theta^\lambda_{\ \rho}\theta^\rho_{\ \nu}\gamma^\nu, \qquad \text{(S.3)} \\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Consequently,

$$\begin{aligned} M^{-1}\gamma^\mu M &= e^{-F}\gamma^\mu e^{+F} \\ &= \gamma^\mu + \big[\gamma^\mu, F\big] + \tfrac{1}{2}\big[\big[\gamma^\mu, F\big], F\big] + \tfrac{1}{6}\big[\big[\big[\gamma^\mu, F\big], F\big], F\big] + \cdots \\ &= \gamma^\mu + \theta^\mu{}_\nu\gamma^\nu + \tfrac{1}{2}\,\theta^\mu{}_\lambda\theta^\lambda{}_\nu\gamma^\nu + \tfrac{1}{6}\,\theta^\mu{}_\lambda\theta^\lambda{}_\rho\theta^\rho{}_\nu\gamma^\nu + \cdots \\ &= L^\mu{}_\nu\gamma^\nu. \end{aligned} \tag{S.4}$$

$\mathcal{Q.E.D.}$

Problem $\mathbf{1}$(c):

$$\begin{aligned} \big\{\gamma^\rho, \gamma^\lambda\gamma^\mu\gamma^\nu\big\} &= 2g^{\rho\lambda}\,\gamma^\mu\gamma^\nu - 2g^{\rho\mu}\,\gamma^\lambda\gamma^\nu + 2g^{\rho\nu}\,\gamma^\lambda\gamma^\mu, \\ \big[\gamma^\rho, \gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu\big] &= 2g^{\rho\kappa}\,\gamma^\lambda\gamma^\mu\gamma^\nu - 2g^{\rho\lambda}\,\gamma^\kappa\gamma^\mu\gamma^\nu + 2g^{\rho\mu}\,\gamma^\kappa\gamma^\lambda\gamma^\nu - 2g^{\rho\nu}\,\gamma^\kappa\gamma^\lambda\gamma^\mu, \\ \big[S^{\rho\sigma}, \gamma^\lambda\gamma^\mu\gamma^\nu\big] &= ig^{\sigma\lambda}\,\gamma^\rho\gamma^\mu\gamma^\nu + ig^{\sigma\mu}\,\gamma^\lambda\gamma^\rho\gamma^\nu + ig^{\sigma\nu}\,\gamma^\lambda\gamma^\mu\gamma^\rho \\ &\qquad - ig^{\rho\lambda}\,\gamma^\sigma\gamma^\mu\gamma^\nu - ig^{\rho\mu}\,\gamma^\lambda\gamma^\sigma\gamma^\nu - ig^{\rho\nu}\,\gamma^\lambda\gamma^\mu\gamma^\sigma. \end{aligned}$$

The algebra is straigtforward.

Problem $\mathbf{1}$(d):

$$\begin{aligned} \gamma^\alpha\,\gamma_\alpha &= \tfrac{1}{2}\{\gamma^\alpha, \gamma^\beta\}g_{\alpha\beta} = g^{\alpha\beta}g_{\alpha\beta} = 4; \\ \gamma^\alpha\gamma^\nu\gamma_\alpha &= 2g^{\alpha\nu}\gamma_\alpha - \gamma^\nu\,\gamma^\alpha\gamma_\alpha = 2\gamma^\nu - \gamma^\nu(4) = -2\gamma^\nu\,; \\ \gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha &= 2g^{\alpha\mu}\gamma^\nu\gamma_\alpha - \gamma^\mu\,\gamma^\alpha\gamma^\nu\gamma_\alpha = 2\gamma^\nu\gamma^\mu - \gamma^\mu\,(-2\gamma^\nu) = 2\{\gamma^\nu, \gamma^\mu\} = 4g^{\mu\nu}\,; \\ \gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\alpha &= 2g^{\alpha\lambda}\gamma^\mu\gamma^\nu\gamma_\alpha - \gamma^\lambda\,\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha = 2\gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\lambda\,(4g^{\mu\nu}) \\ &= 2\big(\gamma^\mu\gamma^\nu - 2g^{\mu\nu}\big)\,\gamma^\lambda = -2\gamma^\nu\gamma^\mu\gamma^\lambda. \end{aligned} \tag{S.5}$$

Problem $\mathbf{1}$(e):

Gauge-covariant derivatives D_μ do not commute with each other: $[D_\mu, D_\nu] = iqF_{\mu\nu}$. Therefore $\big(\gamma^\mu D_\mu\big)^2 \neq D^2$ but rather

$$\big(\gamma^\mu D_\mu\big)^2 = \gamma^\mu\gamma^\nu\,D_\mu D_\nu = (g^{\mu\nu} - 2iS^{\mu\nu})\,D_\mu D_\nu = D^2 - iS^{\mu\nu}\,[D_\mu, D_\nu] = D^2 + qF_{\mu\nu}\,S^{\mu\nu} \tag{S.6}$$

and hence

$$\big(-m - i\gamma^\mu D_\mu\big)\big(-m + i\gamma^\mu D_\mu\big) \;=\; m^2 \;+\; \big(\gamma^\mu D_\mu\big)^2 \;=\; m^2 + \;D^2 \;+\; qF_{\mu\nu}\,S^{\mu\nu}. \tag{S.7}$$

Consequently, the covariant Dirac equation $\big(i\gamma^\mu D_\mu - m\big)\Psi(x) = 0$ implies $\big(m^2 + D^2 + qF_{\mu\nu}S^{\mu\nu}\big)\Psi(x) = 0$.

Problem **2**(a):

First, let us rewrite the Lorentz algebra in terms of 3–vectors $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$:

$$\big[\hat{J}^i, \hat{J}^j\big] \;=\; i\epsilon^{ij\ell}\,\hat{J}^\ell, \quad \big[\hat{J}^i, \hat{K}^j\big] \;=\; i\epsilon^{ij\ell}\,\hat{K}^\ell, \quad \big[\hat{K}^i, \hat{K}^j\big] \;=\; -i\epsilon^{ij\ell}\,\hat{J}^\ell. \tag{S.8}$$

Consequently, for the $\hat{\mathbf{J}}_\pm = \frac{1}{2}\big(\hat{\mathbf{J}} \pm i\hat{\mathbf{K}}\big)$, we have

$$\big[\hat{J}^i_\pm, \hat{J}^j_\pm\big] \;=\; \tfrac{i}{4}\epsilon^{ij\ell}\,\hat{J}^\ell \;\mp\; \tfrac{1}{4}\epsilon^{ij\ell}\,\hat{K}^\ell \;\mp\; \tfrac{1}{4}\epsilon^{ij\ell}\,\hat{K}^\ell \;+\; \tfrac{i}{4}\epsilon^{ij\ell}\,\hat{J}^\ell \;=\; i\epsilon^{ij\ell}\,\hat{J}^\ell_\pm$$

while

$$\big[\hat{J}^i_\pm, \hat{J}^j_\mp\big] \;=\; \tfrac{i}{4}\epsilon^{ij\ell}\,\hat{J}^\ell \;\mp\; \tfrac{1}{4}\epsilon^{ij\ell}\,\hat{K}^\ell \;\pm\; \tfrac{1}{4}\epsilon^{ij\ell}\,\hat{K}^\ell \;-\; \tfrac{i}{4}\epsilon^{ij\ell}\,\hat{J}^\ell \;=\; 0.$$

$\mathcal{Q.E.D.}$

Problem **2**(b):

Let us define two formal 4–vectors of 2×2 matrices $\sigma^\mu \stackrel{\rm def}{=} (1, \vec{\sigma})$ and $\bar{\sigma}^\mu \stackrel{\rm def}{=} (1, -\vec{\sigma})$ where the 3–vector $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ is comprised of the Pauli matrices. In the Weyl convention for the Dirac γ matrices, one has

$$\gamma\mu \;=\; \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{S.9}$$

and hence

$$S^{\mu\nu} \;=\; \begin{pmatrix} \frac{i}{4}\big(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\big) & 0 \\ 0 & \frac{i}{4}\big(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\big) \end{pmatrix} \tag{S.10}$$

In components,

$$S^{0j} \;=\; \begin{pmatrix} -\frac{i}{2}\,\sigma^j & 0 \\ 0 & +\frac{i}{2}\,\sigma^j \end{pmatrix}, \qquad S^{jk} \;=\; \begin{pmatrix} -\frac{i}{4}\,[\sigma^j,\sigma^k] & 0 \\ 0 & -\frac{i}{2}\,[\sigma^j,\sigma^k] \end{pmatrix} \;=\; \epsilon^{jk\ell} \begin{pmatrix} \frac{1}{2}\sigma^\ell & 0 \\ 0 & \frac{1}{2}\sigma^\ell \end{pmatrix} \tag{S.11}$$

and therefore

$$\tfrac{1}{2}\theta_{\mu\nu}\hat{S}^{\mu\nu} \;\equiv\; \tfrac{1}{2}\epsilon^{\ell jk}\,\theta n^\ell S^{jk} \;-\; r^j\,S^{0j} \;=\; \begin{pmatrix} \frac{1}{2}(\vec{\theta}-i\vec{r})\cdot\vec{\sigma} & 0 \\ 0 & \frac{1}{2}(\vec{\theta}+i\vec{r})\cdot\vec{\sigma} \end{pmatrix}. \tag{S.12}$$

Consequently, an infinitesimal Lorentz transformation acts on the Dirac spinor according to

$$\Psi'(x') \;=\; \Psi(x) \;-\; \tfrac{i}{2}\theta_{\mu\nu}\hat{S}^{\mu\nu}\,\Psi(x) = \;\Psi(x) \;+\; \begin{pmatrix} -\frac{i}{2}(\vec{\theta}-i\vec{r})\cdot\vec{\sigma} & 0 \\ 0 & -\frac{i}{2}(\vec{\theta}+i\vec{r})\cdot\vec{\sigma} \end{pmatrix}\Psi(x) \tag{2}$$

Note that the two upper components of the Dirac spinor Ψ do not mix with the two lower components. This makes the 4–component Dirac spinor a *reducible* representation of the continuous Lorentz group. Specifically, the Dirac spinor is a sum of two distinct 2–component irreducible representations, namely the *left-handed Weyl spinor* χ_L and the *right-handed Weyl spinor* χ_R,

$$\Psi(x) \;=\; \begin{pmatrix} \chi_L(x) \\ \chi_R(x) \end{pmatrix}. \tag{S.13}$$

Under finite continuous Lorentz symmetries, the Weyl spinors transform according to

$$\chi'_L(x') \;=\; U_L\,\chi_L(x), \qquad \chi'_R(x') \;=\; U_R\,\chi_R(x), \tag{S.14}$$

where $U_L = \exp\bigl(-\frac{i}{2}(\vec{\theta}-i\vec{r})\cdot\vec{\sigma}\bigr)$ is generally a complex, non-unitary 2×2 matrix of unit determinant and $U_R = \exp\bigl(-\frac{i}{2}(\vec{\theta}+i\vec{r})\cdot\vec{\sigma}\bigr) = \sigma_2 U_L^* \sigma_2$. In the $SL(2,\mathbf{C})$ language of eq. (3), we should indentify U_L as U, hence

$$(\chi_L)'_a(x') \;=\; U_a^{\;b}(\chi_L)_b(x), \qquad (\sigma_2\chi_R)'_{\dot a}(x') \;=\; U^{*\,\dot b}_{\dot a}(\sigma_2\chi_R)_{\dot b}(x). \tag{S.15}$$

Problem **2**(c):

First, according to eq. (4), $U\sigma_\mu U^\dagger = \sigma_\lambda L^\lambda_{\ \mu}(U)$. The hermiticity of the matrices σ_μ and the fact that any hermitian 2×2 matrix is a unique linear combination of the four σ_ν guarantee that the 4×4 matrix $L^\lambda_{\ \mu}(U)$ is well-defined and real.

Next consider the determinant

$$\det\bigl(X_\mu\sigma^\mu\bigr) \;=\; \det\begin{pmatrix} X_0+X_3 & X_1-iX_2 \\ X_1+iX_2 & X_0-X_3 \end{pmatrix} \;=\; (X_0)^2-(X_3)^2-(X_1)^2-(X_2)^2 \;\equiv X^2. \qquad \text{(S.16)}$$

According to eq. (4),

$$X'^2 \;=\; \det\bigl(X'_\mu\sigma^\mu\bigr) \;=\; |\det(U)|^2\,\det\bigl(X_\mu\sigma^\mu\bigr) \;=\; X^2 \qquad \text{(S.17)}$$

because the $SL(2,\mathbf{C})$ matrices U have $\det(U)=1$. Consequently, for *any* 4–vector X^μ, we have $(LX)^2=X^2$, which means that $X^\mu\to L^\mu{}_\nu X^\nu$ is indeed a Lorentz transform.

Finally, consider the group law:

$$\begin{aligned} \sigma_\lambda\, L^\lambda{}_\mu(U_2U_1) \;&=\; (U_2U_1)\sigma_\mu(U_2U_1)^\dagger \;=\; U_2\,\Bigl(U_1\sigma_\mu U_1^\dagger \;=\; \sigma_\nu\,L^\nu{}_\mu(U_1)\Bigr)\,U_2^\dagger \\ &=\; \Bigl(U_2\,\sigma_\nu\,U_2^\dagger\Bigr)\,L^\nu{}_\mu(U_1) \;=\; \sigma_\lambda\,L^\lambda{}_\nu(U_2)\,L^\nu{}_\mu(U_1) \end{aligned} \qquad \text{(S.18)}$$

and hence $L^\lambda{}_\mu(U_2U_1)=L^\lambda{}_\nu(U_2)\,L^\nu{}_\mu(U_1)$ *i.e.*, $L(U_2U_1)=L(U_2)\,L(U_1)$. $\mathcal{Q.E.D.}$

Problem **2**(d):

For $U=\exp\bigl(-\frac{i}{2}\theta\,\mathbf{n}\vec\sigma\bigr)=\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\,\mathbf{n}\vec\sigma$ and $U^\dagger=U^{-1}=cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\,\mathbf{n}\vec\sigma$,
$U\sigma^0U^\dagger=1=\sigma^0$, which means that $L(U)$ is merely a rotation of the 3d space. Specifically,

$$\begin{aligned} \vec\sigma\cdot\mathbf{x}' \;=\; U(\mathbf{x}\vec\sigma)U^\dagger \;&=\; \cos^2\tfrac{\theta}{2} \;-\; \tfrac{i}{2}\sin\theta\,[\mathbf{n}\vec\sigma,\mathbf{x}\vec\sigma] \;+\; \sin^2\tfrac{\theta}{2}\,(\mathbf{n}\vec\sigma)(\mathbf{x}\vec\sigma)(\mathbf{n}\vec\sigma) \\ &=\; \cos^2\tfrac{\theta}{2} \;+\; \sin\theta\,(\mathbf{n}\times\mathbf{x})\cdot\vec\sigma \;+\; \sin^2\tfrac{\theta}{2}\,\bigl(2(\mathbf{nx})(\mathbf{n}\vec\sigma)-(\mathbf{x}\vec\sigma)\bigr) \\ &=\; \vec\sigma\cdot\bigl(\cos\theta\bigl(\mathbf{x}-\mathbf{n}(\mathbf{nx})\bigr) \;+\; \sin\theta\,\mathbf{n}\times\mathbf{x} \;+\; \mathbf{n}(\mathbf{nx})\bigr)\,, \\ \text{thus}\quad \mathbf{x}' \;&=\; \cos\theta\bigl(\mathbf{x}-\mathbf{n}(\mathbf{nx})\bigr) \;+\; \sin\theta\,\mathbf{n}\times\mathbf{x} \;+\; \mathbf{n}(\mathbf{nx}), \end{aligned} \qquad \text{(S.19)}$$

which indeed describes a rotation through angle θ around axis $\mathbf{n}$.

On the other hand, for $U=U^\dagger=\exp\bigl(-\frac{r}{2}\,\mathbf{n}\vec\sigma\bigr)=\cosh\frac{r}{2}-\sinh\frac{r}{2}\,\mathbf{n}\vec\sigma$,

$$\begin{aligned} U\bigl(x^\mu\sigma_\mu\equiv t-\mathbf{x}\vec\sigma\bigr)U^\dagger \;&=\; \cosh^2\tfrac{r}{2}\,(t-\mathbf{x}\vec\sigma) \;-\; \tfrac{1}{2}\sinh r\,\{\mathbf{n}\vec\sigma,t-\mathbf{x}\vec\sigma\} \\ &\qquad\qquad +\; \sinh^2\tfrac{r}{2}\,(\mathbf{n}\vec\sigma)(t-\mathbf{x}\vec\sigma)(\mathbf{n}\vec\sigma) \\ &=\; \cosh^2\tfrac{r}{2}\,(t-\mathbf{x}\vec\sigma) \;-\; \sinh r\,\bigl(t\,\mathbf{n}\vec\sigma-\mathbf{nx}\bigr) \\ &\qquad\qquad +\; \sinh^2\tfrac{r}{2}\,\bigl(t-2(\mathbf{nx})(\mathbf{n}\vec\sigma)+(\mathbf{x}\vec\sigma)\bigr) \\ &=\; \bigl(\cosh r\,t \;+\; \sinh r\,\mathbf{nx}\bigr) \;-\; (\vec\sigma\mathbf{n})\bigl(\sinh r\,t \;+\; \cosh r\,\mathbf{nx}\bigr) \\ &\qquad\qquad -\; \vec\sigma\cdot\bigl(\mathbf{x}-\mathbf{n}(\mathbf{nx})\bigr), \end{aligned} \qquad \text{(S.20)}$$

and therefore,

$$t' \;=\; (\cosh r)\,t \;+\; (\sinh r)\,\mathbf{nx}, \quad \mathbf{x}' \;=\; \mathbf{n}\big((\sinh r)\,t \;+\; (\cosh r)\,\mathbf{nx}\big) \;+\; \big(\mathbf{x}-\mathbf{n}(\mathbf{nx})\big), \tag{S.21}$$

which is precisely the Lorentz boost of rapidity r in the direction $\mathbf{n}$. (The rapidity r is related to the usual parameters of a Lorentz boost according to $\beta=\tanh r$, $\gamma=\cosh r$, $\gamma\beta=\sinh r$. For several boosts in the same directions, the rapidities add up, $r_{\rm tot}=r_1+r_2+\cdots$.) $\mathcal{Q.E.D.}$

Problem **3**(a):

In light of eq. (5),

$$\begin{aligned}\hat\Phi(x'=Lx) \;&=\; \int\frac{d^3\mathbf{p}'}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}'}}\,\Big[e^{-ip'x'}\sqrt{2p'^0}\,\hat a(p') \;+\; e^{+ip'x'}\sqrt{2p'^0}\,\hat a^\dagger(p')\Big]_{p'^0=E_{\mathbf{p}'}}\\ &=\; \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\,\Big[e^{-ipx}\sqrt{2(Lp)^0}\,\hat a(Lp) \;+\; e^{+ipx}\sqrt{2(Lp)^0}\,\hat a^\dagger(Lp)\Big]_{p^0=E_{\mathbf{p}}}\end{aligned} \tag{S.22}$$

where $p'=Lp$ and hence $p'x'=px$ and $\int\frac{d^3\mathbf{p}'}{2E_{\mathbf{p}'}}=\int\frac{d^3\mathbf{p}}{2E_{\mathbf{p}}}$. At the same time, eq. (7) implies

$$\begin{aligned}\hat\Phi(Lx) \;&=\; \hat{\mathcal{D}}(L)\,\hat\Phi(x)\,\hat{\mathcal{D}}^\dagger(L)\\ &=\; \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\,\Big[e^{-ipx}\sqrt{2p^0}\,\hat{\mathcal{D}}(L)\,\hat a(p)\,\hat{\mathcal{D}}^\dagger(L) \;+\; e^{+ipx}\sqrt{2p^0}\,\hat{\mathcal{D}}(L)\,\hat a^\dagger(p)\,\hat{\mathcal{D}}^\dagger(L)\Big]_{p^0=E_{\mathbf{p}}}\end{aligned} \tag{S.23}$$

Since eqs. (S.22) and (S.23) should agree for all x, Fourier transforms of their respective right hand sides should agree for all p, hence

$$\begin{aligned}\sqrt{2p^0}\,\hat{\mathcal{D}}(L)\,\hat a(p)\,\hat{\mathcal{D}}^\dagger(L) \;&=\; \sqrt{2(Lp)^0}\,\hat a(Lp),\\ \sqrt{2p^0}\,\hat{\mathcal{D}}(L)\,\hat a^\dagger(p)\,\hat{\mathcal{D}}^\dagger(L) \;&=\; \sqrt{2(Lp)^0}\,\hat a^\dagger(Lp).\end{aligned} \tag{8.1–2}$$

Consequently,

$$\begin{aligned}\mathcal{D}(L)\,|p\rangle \;&=\; \hat{\mathcal{D}}(L)\left(\sqrt{2p^0}\,\hat a^\dagger(p)\,|0\rangle\right)\\ &=\; \sqrt{2p^0}\,\hat{\mathcal{D}}(L)\,\hat a^\dagger(p)\,\left(|0\rangle=\hat{\mathcal{D}}^\dagger(L)\,|0\rangle\right)\\ &=\; \left(\sqrt{2p^0}\,\hat{\mathcal{D}}(L)\,\hat a^\dagger(p)\,\hat{\mathcal{D}}^\dagger(L)=\sqrt{(Lp)^0}\,\hat a^\dagger(Lp)\right)|0\rangle\\ &=\; |Lp\rangle\,,\end{aligned} \tag{8.3}$$

and likewise

$$
\begin{aligned}
\mathcal{D}(L)\,|p_1,p_2\rangle \;&=\; \hat{\mathcal{D}}(L)\left(\sqrt{(2p_1^0)(2p_2^0)}\,\hat{a}^\dagger(p_1)\hat{a}^\dagger(p_2)\,|0\rangle\right)\\
&=\; \sqrt{(2p_1^0)(2p_2^0)}\,\hat{\mathcal{D}}(L)\,\hat{a}^\dagger(p_1)\hat{a}^\dagger(p_2)\,\left(|0\rangle=\hat{\mathcal{D}}^\dagger(L)\,|0\rangle\right)\\
&=\; \left(\sqrt{2p_1^0}\,\hat{\mathcal{D}}(L)\,\hat{a}^\dagger(p_1)\,\hat{\mathcal{D}}^\dagger(L)\right)\left(\sqrt{2p_2^0}\,\hat{\mathcal{D}}(L)\,\hat{a}^\dagger(p_2)\,\hat{\mathcal{D}}^\dagger(L)\right)|0\rangle\\
&=\; \left(\sqrt{2(Lp_1)^0}\,\hat{a}^\dagger(Lp_1)\right)\left(\sqrt{2(Lp_2)^0}\,\hat{a}^\dagger(Lp_2)\right)|0\rangle\\
&=\; |Lp_1,Lp_2\rangle\,,
\end{aligned}
\tag{8.4}
$$

etc., etc. $\mathcal{Q.E.D.}$

Problem **3**(b):

Using eq. (11) twice, we have

$$
\begin{aligned}
\hat{\mathcal{D}}(L_2)\hat{\mathcal{D}}(L_1)\,\hat{\phi}_a(x)\left(\hat{\mathcal{D}}(L_2)\hat{\mathcal{D}}(L_1)\right)^\dagger \;&=\; \hat{\mathcal{D}}(L_2)\,\left(\hat{\mathcal{D}}(L_1)\,\hat{\phi}_a(x)\,\hat{\mathcal{D}}^\dagger(L_1)\right)\,\hat{\mathcal{D}}^\dagger(L_2)\\
&=\; \hat{\mathcal{D}}(L_2)\,\left(M_a{}^b(L_1^{-1})\,\hat{\phi}_b(L_1x)\right)\,\hat{\mathcal{D}}^\dagger(L_2)\\
&=\; M_a{}^b(L_1^{-1})\,\left(\hat{\mathcal{D}}(L_2)\,\hat{\phi}_b(L_1x)\,\hat{\mathcal{D}}^\dagger(L_2)\right)\\
&=\; M_a{}^b(L_1^{-1})\,M_b{}^c(L_2^{-1})\,\hat{\phi}_c(L_2L_1x).
\end{aligned}
\tag{S.24}
$$

On the other hand,

$$
\hat{\mathcal{D}}(L_2L_1)\,\hat{\phi}_a(x)\,\hat{\mathcal{D}}^\dagger(L_2L_1) \;=\; M_a{}^c\big((L_2L_1)^{-1}=L_1^{-1}L_2^{-1}\big)\,\hat{\phi}_c(L_2L_1x),
\tag{S.25}
$$

which obviously agrees with (S.24) if and only if

$$
M_a{}^c(L_1^{-1}L_2^{-1}) \;=\; M_a{}^b(L_1^{-1})\,M_b{}^c(L_2^{-1})
\tag{S.26}
$$

i.e., if and only if the matrices $M_a{}^b(L)$ form a representation of the Lorentz group.

Problem **3**(b):

Reversing our derivation of eqs. (8.3–4), we see that eqs. (12) require

$$\begin{aligned}
\sqrt{2p^0}\,\hat{\mathcal{D}}(L)\,\hat{a}^\dagger(p,s)\,\hat{\mathcal{D}}^\dagger(L) &= \sum_{s'} C_{s,s'}(L,p)\,\sqrt{2(Lp)^0}\,\hat{a}^\dagger(Lp,s'),\\
\sqrt{2p^0}\,\hat{\mathcal{D}}(L)\,\hat{b}^\dagger(p,s)\,\hat{\mathcal{D}}^\dagger(L) &= \sum_{s'} C_{s,s'}(L,p)\,\sqrt{2(Lp)^0}\,\hat{b}^\dagger(Lp,s'),
\end{aligned} \tag{S.27}$$

and hence, by hermitian conjugation,

$$\begin{aligned}
\sqrt{2p^0}\,\hat{\mathcal{D}}(L)\,\hat{a}(p,s)\,\hat{\mathcal{D}}^\dagger(L) &= \sum_{s'} C^*_{s,s'}(L,p)\,\sqrt{2(Lp)^0}\,\hat{a}(Lp,s'),\\
\sqrt{2p^0}\,\hat{\mathcal{D}}(L)\,\hat{b}(p,s)\,\hat{\mathcal{D}}^\dagger(L) &= \sum_{s'} C^*_{s,s'}(L,p)\,\sqrt{2(Lp)^0}\,\hat{b}(Lp,s').
\end{aligned} \tag{S.28}$$

Consequently, 'sandwiching' both sides of eq. (9) between $\hat{\mathcal{D}}(L)$ and $\hat{\mathcal{D}}^\dagger(L)$ operators gives us

$$\begin{aligned}
\hat{\mathcal{D}}(L)\,\hat{\phi}_a(x)\,\hat{\mathcal{D}}^\dagger(L) &= \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s\Big[e^{-ipx}f_a(p,s)\sqrt{2p^0}\,\Big(\hat{\mathcal{D}}(L)\,\hat{a}(p,s)\,\hat{\mathcal{D}}^\dagger(L)\Big)\\
&\qquad\qquad +\,e^{+ipx}h_a(p,s)\sqrt{2p^0}\,\Big(\hat{\mathcal{D}}(L)\,\hat{b}^\dagger(p,s)\,\hat{\mathcal{D}}^\dagger(L)\Big)\Big]_{p^0=+E_{\mathbf{p}}}\\
&= \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s\Big[e^{-ipx}f_a(p,s)\sqrt{2(Lp)^0}\sum_{s'}C^*_{s,s'}(L,p)\,\hat{a}(Lp,s')\\
&\qquad\qquad +\,e^{+ipx}h_a(p,s)\sqrt{2(Lp)^0}\sum_{s'}C_{s,s'}(L,p)\,\hat{b}^\dagger(Lp,s')\Big]_{p^0=+E_{\mathbf{p}}}\\
&= \int\!\frac{d^3\mathbf{p}'}{(2\pi)^3}\,\frac{1}{2E'_{\mathbf{p}}}\sum_s\Big[e^{-ip'x'}f_a(p,s)\sqrt{2(p')^0}\sum_{s'}C^*_{s,s'}(L,p)\,\hat{a}(p',s')\\
&\qquad\qquad +\,e^{+ip'x'}h_a(p,s)\sqrt{2(p')^0}\sum_{s'}C_{s,s'}(L,p)\,\hat{b}^\dagger(p',s')\Big]_{p'^0=+E_{\mathbf{p}'}}
\end{aligned} \tag{S.29}$$

where on the last line $p'=Lp$ and $x'=Lx$, *cf.* eq. (S.22). Furthermore, according to eq. (11),

$$\hat{\phi}_a(x'=Lx) \;= \sum_b M_a{}^b(L)\,\hat{\mathcal{D}}(L)\,\hat{\phi}_b(x)\,\hat{\mathcal{D}}^\dagger(L) \tag{S.30}$$

and therefore

$$
\begin{aligned}
\hat{\phi}_a(x' = Lx) \;=\; \int \frac{d^3\mathbf{p}'}{(2\pi)^3}\,\frac{1}{2E'_{\mathbf{p}}} \sum_{s'} \Bigg[\; & e^{-ip'x'} \sqrt{2(p')^0}\,\hat{a}(p's') \\
& \times \sum_b M_a^{\;b}(L) \sum_s C^*_{s,s'}(L,(L^{-1}p'))\, f_b((L^{-1}p'),s) \\
& + e^{+ip'x'} \sqrt{2(p')^0}\,\hat{b}^\dagger(p's') \\
& \times \sum_b M_a^{\;b}(L) \sum_s C_{s,s'}(L,(L^{-1}p'))\, h_b((L^{-1}p'),s) \Bigg]_{p'^0=+E_{\mathbf{p}'}}
\end{aligned}
\tag{S.31}
$$

On the other hand, a simple change of variables $x \to x'$, $p \to p'$ in eq. (9) gives

$$
\begin{aligned}
\hat{\phi}_a(x' = Lx) \;=\; \int \frac{d^3\mathbf{p}'}{(2\pi)^3}\,\frac{1}{2E'_{\mathbf{p}}} \sum_{s'} \Big[& e^{-ip'x'} f_a(p',s') \sqrt{2(p')^0}\,\hat{a}(p',s') \\
& + e^{+ip'x'} h_a(p',s') \sqrt{2(p')^0}\,\hat{b}^\dagger(p',s') \Big]_{p^0=+E_{\mathbf{p}}}
\end{aligned}
\tag{S.32}
$$

Comparing the right hand sides of eqs. (S.31) and (S.32) we see similar operators accompanied by similar exponentials, so all we need now is similar coefficients, thus

$$
\begin{aligned}
f_a(p' = Lp, s') \;&=\; \sum_b M_a^{\;b}(L) \sum_s C^*_{s,s'}(L,p)\, f_b(p,s), \\
h_a(p' = Lp, s') \;&=\; \sum_b M_a^{\;b}(L) \sum_s C_{s,s'}(L,p)\, h_b(p,s),
\end{aligned}
\tag{13}
$$

$\mathcal{Q.E.D.}$

As an example, consider a massive vector field $\hat{A}^\mu(x)$ which we have (in previous exercises) written in the form (9) where $\hat{b}^\dagger(p,\lambda) = \hat{a}^\dagger(p,\lambda)$ (due to hermiticity of $\hat{A}^\mu(x)$) and $f^\mu(p,\lambda)$ plays the role of $f_a(p,s)$ (as well as $h^*_a(p,s)$). Consequently, $f^\mu(p,\lambda)$ indeed transform according to eq. (13) where $M^\mu_{\;\nu}(L) = L^\mu_{\;\nu}$, as appropriate for the vector representation of the Lorentz group, while the matrix $C_{\lambda,\lambda'}$ rotates the helicity states into each other.

Similarly the Dirac spinors $u(p,s)$ and $v(p,s)$ also transform according to eqs. (13) where $M(L)$ — *cf.* problems **1**(b) and **2** — is a 4×4 matrix representing L in the spinor representation of the Lorentz group while the $C_{s,s'}$ matrices acts on the 3D spinors ξ_s used for construction of the Dirac spinors $u(p,s)$ and $v(p,s)$.