Problem **1**(a):

$\gamma^\mu\gamma^\nu = \pm\gamma^\nu\gamma^\mu$ where the sign is '+' for $\mu=\nu$ and '−' otherwise. Hence for any product Γ of the γ matrices, $\gamma^\mu\Gamma = (-1)^{n_\mu}\Gamma\gamma^\mu$ where n_μ is the number of $\gamma^{\nu\neq\mu}$ factors of Γ. For $\Gamma=\gamma^5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, $n_\mu=3$ for any $\mu=0,1,2,3$; thus $\gamma^\mu\gamma^5=-\gamma^5\gamma^\mu$.

Problem **1**(b):

First,

$$\begin{aligned}\left(\gamma^5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3\right)^\dagger &= -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger \;=\; +i\gamma^3\gamma^2\gamma^1\gamma^0\\ &= +i((\gamma^3\gamma^2)\gamma^1)\gamma^0 \;=\; (-1)^3 i\,\gamma^0((\gamma^3\gamma^2)\gamma^1)\\ &= (-1)^{3+2} i\,\gamma^0(\gamma^1(\gamma^3\gamma^2)) \;=\; (-1)^{3+2+1} i\,\gamma^0(\gamma^1(\gamma^2\gamma^3))\\ &= +i\gamma^0\gamma^1\gamma^2\gamma^3 \;\equiv\; +\gamma^5.\end{aligned}\tag{S.1}$$

Second,

$$\begin{aligned}(\gamma^5)^2 &= \gamma^5(\gamma^5)^\dagger = (i\gamma^0\gamma^1\gamma^2\gamma^3)(i\gamma^3\gamma^2\gamma^1\gamma^0) = -\gamma^0\gamma^1\gamma^2(\gamma^3\gamma^3)\gamma^2\gamma^1\gamma^0\\ &= +\gamma^0\gamma^1(\gamma^2\gamma^2)\gamma^1\gamma^0 = -\gamma^0(\gamma^1\gamma^1)\gamma^0 = +\gamma^0\gamma^0 = +1.\end{aligned}\tag{S.2}$$

Problem **1**(c):

Any four distinct γ^κ, γ^λ, γ^μ, and γ^ν are γ^0, γ^1, γ^2, and γ^3 in some order. They all anticommute with each other, hence $\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu = \epsilon^{\kappa\lambda\mu\nu}\gamma^0\gamma^1\gamma^2\gamma^3 \equiv -i\epsilon^{\kappa\lambda\mu\nu}\gamma^5$. The rest is obvious.

Problem **1**(d):

$$\begin{aligned}i\epsilon^{\kappa\lambda\mu\nu}\,\gamma_\kappa\gamma^5 &= \gamma_\kappa\,\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]}\\ &= \tfrac14\,\gamma_\kappa\left(\gamma^\kappa\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} - \gamma^{[\lambda)}\gamma^\kappa\gamma^{(\mu}\gamma^{\nu]} + \gamma^{[\lambda}\gamma^{\mu)}\gamma^\kappa\gamma^{(\nu]} - \gamma^{[\lambda}\gamma^\mu\gamma^{\nu]}\gamma^\kappa\right)\\ &= \tfrac14\left(4\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} + 2\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} + 4g^{[\lambda\mu}\gamma^{\nu]} + 2\gamma^{[\nu}\gamma^\mu\gamma^{\lambda]}\right)\\ &= \tfrac14(4+2+0-2)\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = \gamma^{[\lambda}\gamma^\mu\gamma^{\nu]}.\end{aligned}\tag{S.3}$$

Problem **1**(e):

Proof by inspection: In the Weyl basis, the 16 matrices are

$$\mathbf{1}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \gamma^\mu=\begin{pmatrix}0&\sigma^\mu\\ \bar\sigma^\mu&0\end{pmatrix},\quad \gamma^{[\mu}\gamma^{\nu]}=\begin{pmatrix}\sigma^{[\mu}\bar\sigma^{\nu]}&0\\0&\bar\sigma^{[\mu}\sigma^{\nu]}\end{pmatrix},\quad \gamma^5\gamma^\mu=\begin{pmatrix}0&-\sigma^\mu\\ \bar\sigma^\mu&0\end{pmatrix},\quad \gamma^5=\begin{pmatrix}-1&0\\0&+1\end{pmatrix},\tag{S.4}$$

and their linear independence is self-evident. Since there are only 16 independent 4×4 matrices altogether, any such matrix Γ is a linear combination of the matrices (S.4). $\mathcal{Q.E.D.}$

Algebraic Proof: Without making any assumption about the matrix form of the γ^μ operators, let us consider the Clifford algebra $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}$. Because of these anticommutation relations, one may re-order any product of the γ's as $\pm\gamma^0\cdots\gamma^0\,\gamma^1\cdots\gamma^1\,\gamma^2\cdots\gamma^2\,\gamma^3\cdots\gamma^3$ and then further simplify it to $\pm(\gamma^0 \text{ or } 1)\times(\gamma^1 \text{ or } 1)\times(\gamma^2 \text{ or } 1)\times(\gamma^3 \text{ or } 1)$. The net result is (up to a sign or $\pm i$ factor) one of the 16 operators 1, γ^μ, $\gamma^{[\mu}\gamma^{\nu]}$, $\gamma^5\gamma^\mu$ (*cf.* (d)) or γ^5 (*cf.* (c)). Consequently, any operator Γ algebraically constructed of the γ^μ's is a linear combination of these 16 operators.

Incidentally, the algebraic argument explains why the γ^μ (and hence all their products) should be realized as 4×4 matrices since any lesser matrix size would not accomodate 16 independent products. That is, the γ's are 4×4 matrices in four spacetime dimensions; different dimensions call for different matrix sizes. Specifically, in spacetimes of *even* dimensions d, there are 2^d independent products of the γ operators, so we need matrices of size $2^{d/2}\times2^{d/2}$: 2×2 in two dimensions, 4×4 in four, 8×8 in six, 16×16 in eight, 32×32 in ten, *etc.*,*etc.*.

In odd dimensions, there are only 2^{d-1} independent operators because $\gamma^{d+1}\equiv(i)\gamma^0\gamma^1\cdots\gamma^{d-1}$ — the analogue of the γ^5 operator in 4d — commutes rather than anticommutes with all the γ^μ and hence with the whole algebra. Consequently, one has two distinct representations of the Clifford algebra — one with $\gamma^{d+1}=+1$ and one with $\gamma^{d+1}=-1$ — but in each representation there are only 2^{d-1} independent operator products, which call for the matrix size of $2^{(d-1)/2}\times2^{(d-1)/2}$. For example, in three spacetime dimensions (two space, one time), can take $(\gamma^0,\gamma^1,\gamma^2)=(\sigma_3,i\sigma_1,i\sigma_2)$ for $\gamma^4\equiv i\gamma^0\gamma^1\gamma^2=+1$ or $(\gamma^0,\gamma^1,\gamma^2)=(\sigma_3,i\sigma_1,-i\sigma_2)$ for $\gamma^4=-1$, but in both cases we have 2×2 matrices.

Problem **1**(f):

Under a continuous Lorentz symmetry $x\mapsto x'=Lx$, the Dirac spinor field and its conjugate transfrom according to

$$\Psi'(x') \;=\; M(L)\Psi(x = L^{-1}x'), \qquad \overline{\Psi}'(x') \;=\; \overline{\Psi}(x = L^{-1}x')M^{-1}(L), \tag{S.5}$$

hence any bilinear $\overline{\Psi}\Gamma\Psi$ transforms according to

$$\overline{\Psi}'(x')\Gamma\Psi(x') \;=\; \overline{\Psi}(x)\Gamma'\Psi(x) \tag{S.6}$$

where

$$\Gamma' = M^{-1}(L)\Gamma M(L). \tag{S.7}$$

Obviously, for $\Gamma = 1$, $\Gamma' = M^{-1}M = 1$. According to previous homeworks, for $\Gamma = \gamma^\mu$, $\Gamma' = M^{-1}\gamma^\mu M = L^\mu_{\ \nu}\gamma^\nu$. Similarly, $M^{-1}\gamma^\mu\gamma^\nu M = L^\mu_{\ \alpha}\gamma^\alpha L^\nu_{\ \beta}\gamma^\beta$ and hence for $\Gamma = \gamma^{[\mu}\gamma^{\nu]}$, $\Gamma' = L^\mu_{\ \alpha}L^\nu_{\ \beta}\gamma^{[\alpha}\gamma^{\beta]}$. Consequently,

$$S'(x') \;=\; S(x), \qquad V'^\mu(x') \;=\; L^\mu_{\ \nu}V^\nu(x), \qquad T'^{\mu\nu}(x') \;=\; L^\mu_{\ \alpha}L^\nu_{\ \beta}T^{\alpha\beta}(x), \tag{S.8}$$

which makes S a Lorentz scalar, V^μ a Lorentz vector and $T^{\mu\nu}$ a Lorentz tensor (with two antisymmetric indices).

The γ^5 matrix commutes with even products of the γ^μ matrices such as $\gamma^\mu\gamma^\nu$, hence it commutes with all $S^{\mu\nu}$ and therefore with $M(L) = \exp\bigl(-\frac{i}{2}\theta_{\mu\nu}S^{\mu\nu}\bigr)$. Consequently, for $\Gamma = \gamma^5$, $\Gamma' = M^{-1}\gamma^5 M = \gamma^5$ while for $\Gamma = \gamma^5\gamma^\mu$, $\Gamma' = M^{-1}\gamma^5\gamma^\mu M = \gamma^5 M^{-1}\gamma^\mu M = \gamma^5(L^\mu_{\ \nu}\gamma^\nu) = L^\mu_{\ \nu}(\gamma^5\gamma^\nu)$. Therefore,

$$P'(x') \;=\; P(x), \qquad A'^\mu(x') \;=\; L^\mu_{\ \nu}A^\nu(x), \tag{S.9}$$

which makes P a Lorentz scalar and A a Lorentz vector. $\mathcal{Q.E.D.}$

Problem **2**(a):

Given $\Psi'(\mathbf{x}', t) = \pm\gamma^0\Psi(\mathbf{x} = -\mathbf{x}', t' = t)$, we have

$$\begin{aligned}(i\,\partial\!\!\!/' - m)\Psi'(x') \;&\equiv\; (i\gamma^0\partial_0 + i\vec{\gamma}\cdot\nabla' - m)(\pm\gamma^0)\Psi(\mathbf{x}', t) \;=\; (\pm\gamma^0)(i\gamma^0\partial_0 - i\vec{\gamma}\cdot\nabla' - m)\Psi(\mathbf{x}', t)\\ &=\; (\pm\gamma^0)(i\gamma^0\partial_0 + i\vec{\gamma}\cdot\nabla - m)\Psi(-\mathbf{x}, t)\\ &\equiv\; (\pm\gamma^0)(i\,\partial\!\!\!/ - m)\Psi\Big|_{x'}.\end{aligned} \tag{S.10}$$

Problem **2**(b):

Parity properties of the Dirac bilinears (1) follow from the commutation relations of the 16 operators (1e) with the γ^0. It is easy to verify that the 1, γ^0, $\gamma^{[i}\gamma^{j]}$ and $\gamma^5\gamma^i$ commute with the γ^0 while the γ^i, $\gamma^0\gamma^i$, $\gamma^5\gamma^0$ and γ^5 anticommute with the γ^0. Consequently,

- the S, V^0, T^{ij} and A^i remain invariant under parity, while
- the V^i, T^{0i}, A^0 and P change their signs.

In three-dimensional terms, this means that S and V^0 are true scalars, P and A^0 are pseudoscalars, $\mathbf{V}$ is a true or polar vector, $\mathbf{A}$ is a pseudovector or axial vector, and the tensor T contains one true vector T^{0i} and one axial vector $\frac{1}{2}\epsilon^{ijk}T^{jk}$. In space-time terms, we call S a (Lorentz) (true) scalar, P a (Lorentz) pseudoscalar, V^μ a (Lorentz) (true) vector and A^μ an (Lorentz) axial vector. Pedantically speaking, $T^{\mu\nu}$ is a Lorentz true tensor while $\widetilde{T}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}T_{\alpha\beta}$ is a Lorentz pseudotensor, but few people are that pedantic.

Problem **3**(a):

The overall statistics of the operator product $\hat{B}\hat{C}$ corresponds to $(-1)^{A(BC)} = (-1)^{AB}(-1)^{AC}$. Therefore,

$$\begin{aligned} [\hat{A}, \hat{B}\hat{C}\} &\stackrel{\text{def}}{=} \hat{A}\hat{B}\hat{C} \,-\, (-1)^{AB}(-1)^{AC}\hat{B}\hat{C}\hat{A} \\ &= \left(\hat{A}\hat{B} - (-1)^{AB}\hat{B}\hat{A}\right)\hat{C} \,+\, (-1)^{AB}\hat{B}\left(\hat{A}\hat{C} - (-1)^{AC}\hat{C}\hat{A}\right) \\ &= [\hat{A}, \hat{B}\}\hat{C} \,+\, (-1)^{AB}\hat{B}[\hat{A}, \hat{C}\}. \end{aligned} \tag{S.11}$$

Likewise,

$$\begin{aligned} [\hat{A}\hat{B}, \hat{C}\} &= \hat{A}\hat{B}\hat{C} \,-\, (-1)^{AC}(-1)^{BC}\hat{C}\hat{A}\hat{B} \\ &= \hat{A}\left(\hat{B}\hat{C} - (-1)^{BC}\hat{C}\hat{B}\right) \,+\, (-1)^{BC}\left(\hat{A}\hat{C} - (-1)^{AC}\hat{C}\hat{A}\right)\hat{B} \\ &= \hat{A}[\hat{B}, \hat{C}\} \,+\, (-1)^{BC}[\hat{A}, \hat{C}\}\hat{B}. \end{aligned} \tag{S.12}$$

Problem **3**(b):

$$\begin{aligned} [\hat{A}\hat{B}, \hat{C}\hat{D}\} &= \hat{A}[\hat{B}, \hat{C}\hat{D}\} \,+\, (-1)^{BC}(-1)^{BD}[\hat{A}, \hat{C}\hat{D}\}\hat{B} \\ &= \hat{A}[\hat{B}, \hat{C}\}\hat{D} \,+\, (-1)^{BC}\hat{A}\hat{C}[\hat{B}, \hat{D}\} \\ &\quad + (-1)^{BC}(-1)^{BD}[\hat{A}, \hat{C}\}\hat{D}\hat{B} \,+\, (-1)^{AC}(-1)^{BC}(-1)^{BD}\hat{C}[\hat{A}, \hat{D}\}\hat{B}. \end{aligned} \tag{S.13}$$

Problem **3**(c):

$$\begin{aligned}
(-1)^{CA}[\hat{A},[\hat{B},\hat{C}\}\} \;&=\; (-1)^{CA}\hat{A}\hat{B}\hat{C} \;-\; (-1)^{BC}(-1)^{CA}\hat{A}\hat{C}\hat{B} \\
&\quad -\; (-1)^{AB}\hat{B}\hat{C}\hat{A} \;+\; (-1)^{AB}(-1)^{BC}\hat{C}\hat{B}\hat{A}, \\
(-1)^{BC}[\hat{C},[\hat{A},\hat{B}\}\} \;&=\; (-1)^{BC}\hat{C}\hat{A}\hat{B} \;-\; (-1)^{AB}(-1)^{BC}\hat{C}\hat{B}\hat{A} \\
&\quad -\; (-1)^{CA}\hat{A}\hat{B}\hat{C} \;+\; (-1)^{CA}(-1)^{AB}\hat{B}\hat{A}\hat{C}, \\
(-1)^{AB}[\hat{B},[\hat{C},\hat{A}\}\} \;&=\; (-1)^{AB}\hat{B}\hat{C}\hat{A} \;-\; (-1)^{CA}(-1)^{AB}\hat{B}\hat{A}\hat{C} \\
&\quad -\; (-1)^{BC}\hat{C}\hat{A}\hat{B} \;+\; (-1)^{BC}(-1)^{CA}\hat{A}\hat{C}\hat{B}.
\end{aligned} \tag{S.14}$$

Upon adding these three equations together, their right hand sides cancel out while the left hand sides add up to the Jacobi identity (4).

Problem **4**(a):

Using the Leibnitz rules (S.11) and (S.12) and the anticommutation relations (6), the calculation is straightforward.

$$\begin{aligned}
[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma] \;&=\; \delta_{\beta\gamma}\,\hat{a}^\dagger_\alpha\,, \\
[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}_\delta] \;&=\; -\delta_{\alpha\delta}\,\hat{a}_\beta\,, \\
[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta] \;&=\; \delta_{\beta\gamma}\,\hat{a}^\dagger_\alpha \hat{a}_\delta \;-\; \delta_{\alpha\delta}\,\hat{a}^\dagger_\gamma \hat{a}_\beta\,.
\end{aligned} \tag{S.15}$$

Problem **4**(b):

According to eq. (S.15), the commutator $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta]$ has exactly the same form as its bosonic counterpart. Hence, the proof of $[\hat{A},\hat{B}] = \hat{C}$ proceeds exactly as in the bosonic case, *cf.* homework set#1 (problem 3(b)).

Problem **4**(c):

Using the Leibnitz rules and eqs. (S.15),

$$[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta] \;=\; \delta_{\nu\alpha}\hat{a}^\dagger_\mu \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta \;+\; \delta_{\nu\beta}\hat{a}^\dagger_\mu \hat{a}^\dagger_\alpha \hat{a}_\gamma \hat{a}_\delta \;-\; \delta_{\mu\gamma}\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\nu \hat{a}_\delta \;-\; \delta_{\mu\delta}\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\nu\,. \tag{S.16}$$

Problem **4**(d):

Again, we have a fermionic analogue to the bosonic second-quantized operators we studied in homework set#1 (problem 3(d)). Given eqs. (7) and (S.16) (in which we exchange $\gamma \leftrightarrow \delta$), we have

$$
\begin{aligned}
\left[\hat{A}, \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\delta \hat{a}_\gamma\right] &= \sum_{\mu,\nu} \langle\mu|\,\hat{A}_1\,|\nu\rangle\,\left[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\delta \hat{a}_\gamma\right] \\
&= \sum_{\mu} \langle\mu|\,\hat{A}_1\,|\alpha\rangle\ \hat{a}^\dagger_\mu \hat{a}^\dagger_\beta \hat{a}_\delta \hat{a}_\gamma\ + \sum_{\mu} \langle\mu|\,\hat{A}_1\,|\beta\rangle\ \hat{a}^\dagger_\alpha \hat{a}^\dagger_\mu \hat{a}_\delta \hat{a}_\gamma \\
&\quad - \sum_{\nu} \langle\delta|\,\hat{A}_1\,|\nu\rangle\ \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\nu \hat{a}_\gamma\ - \sum_{\nu} \langle\gamma|\,\hat{A}_1\,|\nu\rangle\ \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\delta \hat{a}_\nu
\end{aligned}
\tag{S.17}
$$

and consequently, in light of eq. (8),

$$
\begin{aligned}
\left[\hat{A}, \hat{B}\right] &= \sum_{\alpha,\beta,\gamma,\delta} \langle\alpha\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\delta\rangle\ \Bigg[\sum_{\mu} \langle\mu|\,\hat{A}_1\,|\alpha\rangle\ \hat{a}^\dagger_\mu \hat{a}^\dagger_\beta \hat{a}_\delta \hat{a}_\gamma\ + \sum_{\mu} \langle\mu|\,\hat{A}_1\,|\beta\rangle\ \hat{a}^\dagger_\alpha \hat{a}^\dagger_\mu \hat{a}_\delta \hat{a}_\gamma \\
&\qquad - \sum_{\nu} \langle\delta|\,\hat{A}_1\,|\nu\rangle\ \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\nu \hat{a}_\gamma\ - \sum_{\nu} \langle\gamma|\,\hat{A}_1\,|\nu\rangle\ \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\delta \hat{a}_\nu\Bigg] \\
&= \sum_{\mu,\beta,\gamma,\delta} \langle\mu\otimes\beta|\,\hat{A}_1(1^{\underline{\text{st}}})\hat{B}_2\,|\gamma\otimes\delta\rangle\,\hat{a}^\dagger_\mu \hat{a}^\dagger_\beta \hat{a}_\delta \hat{a}_\gamma \\
&\quad + \sum_{\alpha,\mu,\gamma,\delta} \langle\alpha\otimes\mu|\,\hat{A}_1(2^{\underline{\text{nd}}})\hat{B}_2\,|\gamma\otimes\delta\rangle\,\hat{a}^\dagger_\alpha \hat{a}^\dagger_\mu \hat{a}_\delta \hat{a}_\gamma \\
&\quad - \sum_{\alpha,\beta,\gamma,\nu} \langle\alpha\otimes\beta|\,\hat{B}_2\hat{A}_1(2^{\underline{\text{nd}}})\,|\gamma\otimes\nu\rangle\,\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\nu \hat{a}_\gamma \\
&\quad - \sum_{\alpha,\beta,\nu,\delta} \langle\alpha\otimes\beta|\,\hat{B}_2\hat{A}_1(1^{\underline{\text{st}}})\,|\nu\otimes\delta\rangle\,\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\delta \hat{a}_\nu \\
&\langle\!\langle\text{renaming indices}\rangle\!\rangle \\
&= \sum_{\alpha,\beta,\gamma,\delta} \langle\alpha\otimes\beta|\left[\left(A_1(1^{\underline{\text{st}}}) + A_1(2^{\underline{\text{nd}}})\right), \hat{B}_2\right]|\gamma\otimes\delta\rangle \times \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\delta \hat{a}_\gamma \\
&= \sum_{\alpha,\beta,\gamma,\delta} \langle\alpha\otimes\beta|\,\hat{C}_2\,|\gamma\otimes\delta\rangle \times \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\delta \hat{a}_\gamma\ \equiv\ \hat{C}.
\end{aligned}
$$

Q. E. D.