Problem **1**(a):

In the two-component spinor notations

$$\Psi\;=\;\begin{pmatrix}\chi\\ -\sigma_2\varphi^*\end{pmatrix},\qquad \overline{\Psi}\;=\;\left(-\varphi^\top\sigma_2,\chi^\dagger\right),\qquad \gamma^\mu\;=\;\begin{pmatrix}0&\sigma^\mu\\ \bar\sigma^\mu&0\end{pmatrix}.\tag{S.1}$$

Consequently, the Dirac Lagrangian becomes

$$\mathcal{L}_D\;\equiv\;\overline{\Psi}(i\gamma^\mu\partial_\mu-m)\Psi\;=\;i\chi^\dagger\bar\sigma^\mu\partial_\mu\chi\;+\;i\varphi^\top\sigma_2\sigma^\mu\sigma_2\partial_\mu\varphi^*\;+\;m\left(\varphi^\top\sigma_2\chi\;+\;\chi^\dagger\sigma_2\varphi^*\right).\tag{S.2}$$

The second term here looks unduly complicated, so let us transpose it. Because of the fermionic nature of the φ and φ^* fields, there is a sign change when we change their order, thus

$$\varphi^\top\sigma_2\sigma^\mu\sigma_2\partial_\mu\varphi^*\;=\;-(\partial_\mu\varphi^\dagger)(\sigma_2\sigma^\mu\sigma_2)^\top\varphi\;=\;-(\partial_\mu\varphi^\dagger)\bar\sigma^\mu\varphi\;=\;+\varphi^\dagger\bar\sigma^\mu\partial_\mu\varphi\;+\;\partial_\mu(\cdots).\tag{S.3}$$

Substituting this formula into eq. (S.2) immediately gives us eq. (2), modulo a total derivative.

Problem **1**(b):

As discussed in class, the **CP** — Charge conjugation + Parity — symmetry transforms the Weyl spinor fields according to

$$\widehat{\mathcal{CP}}\chi(\mathbf{x},t)\widehat{\mathcal{CP}}\;=\;\pm\sigma_2\chi^*(-\mathbf{x},t),\qquad \widehat{\mathcal{CP}}\varphi(\mathbf{x},t)\widehat{\mathcal{CP}}\;=\;\mp\sigma_2\varphi^*(-\mathbf{x},t).\tag{S.4}$$

Consequently, the kinetic term for the φ spinor — the second term on the right hand side of eq. (2) — becomes

$$\begin{aligned}\widehat{\mathcal{CP}}\left(i\varphi^\dagger\bar\sigma^\mu\partial_\mu\varphi\right)(\mathbf{x},t)\widehat{\mathcal{CP}}\;&=\;i\Big(\mp\varphi^\top(-\mathbf{x},t)\sigma_2\Big)\bar\sigma^\mu\Big(\mp\sigma_2\partial_\mu\varphi^*(-\mathbf{x},t)\Big)\\ &=\;i\Big(\varphi^\top\sigma_2\bar\sigma^0\sigma_2\partial_0\varphi^*\;-\;\varphi^\top\sigma_2\bar\sigma^i\sigma_2\partial_i\varphi^*\Big)_{(-\mathbf{x},t)}\\ &=\;i\Big(\varphi^\top\sigma_2\sigma^0\sigma_2\partial_0\varphi^*\;+\;\varphi^\top\sigma_2\sigma^i\sigma_2\partial_i\varphi^*\Big)_{(-\mathbf{x},t)}\\ &\equiv\;\Big(i\varphi^\top\sigma_2\sigma^\mu\sigma_2\partial_\mu\varphi^*\Big)_{(-\mathbf{x},t)}\\ &=\;\Big(i\varphi^\dagger\bar\sigma^\mu\partial_\mu\varphi\Big)_{(-\mathbf{x},t)}\end{aligned}\tag{S.5}$$

where the last equality is eq. (S.3). In other words, the kinetic Lagrangian term for the φ spinor is **CP** invariant — and of course the kinetic term for the χ spinor is **CP** invariant in exactly same way. As to the mass term, we have

$$\widehat{\mathcal{CP}}\Big(\varphi^\top\sigma_2\chi\Big)\widehat{\mathcal{CP}} \;=\; (\mp\sigma^2\varphi^*)^\top\sigma_2(\pm\sigma_2\chi^*) \;=\; -\varphi^\dagger\sigma_2^\top\sigma_2\sigma_2\chi^* \;=\; +\varphi^\dagger\sigma_2\chi^* \tag{S.6}$$

and likewise

$$\widehat{\mathcal{CP}}\Big(\varphi^\dagger\sigma_2\chi^*\Big)\widehat{\mathcal{CP}} \;=\; (\mp\sigma^2\varphi^*)^\dagger\sigma_2(\pm\sigma_2\chi^*)^* \;=\; -\varphi^\top\sigma_2^\dagger\sigma_2\sigma_2^*\chi \;=\; +\varphi^\top\sigma_2\chi\,, \tag{S.7}$$

so the complete mass term

$$m\Big(\varphi^\top\sigma_2\chi\,+\,\varphi^\dagger\sigma_2\chi^*\Big) \tag{S.8}$$

is indeed **CP** invariant.

Problem **2**(a):

Charge conjugation acts on a Dirac spinor field according to $\hat{\mathcal{C}}\hat{\Psi}\hat{\mathcal{C}} = \pm\gamma^2\hat{\Psi}^*$. Consequently,

$$\hat{\mathcal{C}}\hat{\overline{\Psi}}\hat{\mathcal{C}} \;=\; (\hat{\mathcal{C}}\hat{\Psi}\hat{\mathcal{C}})^\dagger\gamma^0 \;=\; \mp\Psi^\top\gamma^2\gamma^0 \tag{S.9}$$

and hence

$$\hat{\mathcal{C}}\hat{\overline{\Psi}}\Gamma\hat{\Psi}\hat{\mathcal{C}} \;=\; \hat{\mathcal{C}}\hat{\overline{\Psi}}\hat{\mathcal{C}}\Gamma\hat{\mathcal{C}}\hat{\Psi}\hat{\mathcal{C}} \;=\; -\hat{\Psi}^\top\gamma^2\gamma^0\Gamma\gamma^2\hat{\Psi}^* \;=\; +\hat{\Psi}^\dagger(\gamma^2\gamma^0\Gamma\gamma^2)^\top\hat{\Psi} \;=\; \hat{\overline{\Psi}}\gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2\hat{\Psi} \;\equiv\; \hat{\overline{\Psi}}\Gamma^c\hat{\Psi}. \tag{S.10}$$

Problem **2**(b):

By inspection, $\mathbf{1}^c \equiv \gamma^0\gamma^2\gamma^0\gamma^2 = +\mathbf{1}$. The γ_5 matrix is symmetric and commutes with the $\gamma^0\gamma^2$, hence $\gamma_5^c = +\gamma_5$. Among the four γ_μ matrices, the γ_1 and γ_3 are anti-symmetric and commute with the $\gamma^0\gamma^2$ while the γ_0 and γ_2 are symmetric but anti-commute with the $\gamma^0\gamma^2$; hence, for all four γ_μ, $\gamma_\mu^c = -\gamma_\mu$. Finally, because of the transposition involved, $(\gamma_\mu\gamma_\nu)^c = \gamma_\nu^c\gamma_\mu^c = +\gamma_\nu\gamma_\mu$, thus $\gamma_{\mu\nu}^c = +\gamma_{\nu\mu} = -\gamma_{\mu\nu}$. Likewise, $(\gamma_5\gamma_\mu)^c = \gamma_\mu^c\gamma_5^c = -\gamma_\mu\gamma_5 = +\gamma_5\gamma_\mu$.

To summarize, the scalar S, the pseudoscalar P and the axial vector A_μ are C-even while the vector V_μ and the tensor $T_{\mu\nu}$ are C-odd.

Problem **2**(c):

A fermion-antifermion bound state with a wave-function $\psi(\mathbf{p},s_f,s_a)$ is created from the vacuum by the operator

$$\hat{B} = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\sum_{s_f,s_a}\psi(\mathbf{p},s_f,s_a)\,\hat{a}^\dagger_{\mathbf{p},s_f}\hat{b}^\dagger_{-\mathbf{p},s_a}\,. \tag{S.11}$$

Charge-conjugating this operator yields

$$\begin{aligned}\hat{\mathcal{C}}\hat{B}\hat{\mathcal{C}} &= \int\frac{d^3\mathbf{p}}{(2\pi)^3}\sum_{s_f,s_a}\psi(\mathbf{p},s_f,s_a)\,\hat{b}^\dagger_{\mathbf{p},s_f}\hat{a}^\dagger_{-\mathbf{p},s_a} \\ &= -\int\frac{d^3\mathbf{p}}{(2\pi)^3}\sum_{s_f,s_a}\psi(\mathbf{p},s_f,s_a)\,\hat{a}^\dagger_{-\mathbf{p},s_a}\hat{b}^\dagger_{\mathbf{p},s_f} = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\sum_{s_f,s_a}[-\psi(-\mathbf{p},s_a,s_f)]\,\hat{a}^\dagger_{\mathbf{p},s_f}\hat{b}^\dagger_{-\mathbf{p},s_a}\,.\end{aligned} \tag{S.12}$$

Given the net spin S and orbital angular momentum L of a bound state, we have $\psi_{\text{spin}}(s_f,s_a) = -(-1)^S\psi_{\text{spin}}(s_a,s_f)$ and $\psi_{\text{space}}(-\mathbf{p}) = (-1)^L\psi_{\text{space}}(+\mathbf{p})$, hence altogether $[-\psi(-\mathbf{p},s_a,s_f)] = (-1)^S(-1)^L\psi(\mathbf{p},s_f,s_a)$. Consequently, $\hat{\mathcal{C}}\hat{B}\,|0\rangle = (-1)^S(-1)^L\hat{B}\,|0\rangle$ *i.e.*, the bound state $\hat{B}\,|0\rangle$ has $C = (-1)^S(-1)^L$. $\mathcal{Q.E.D.}$

Quantum electrodynamics is invariant under the charge conjugagion, hence C is conserved in electromagnetic processes. The photons are C-odd, hence the electromagnetic decay (or internal annihilation) of a C-even bound state results in an even number of photons while for the C-odd states' decay the number of resulting photons should be odd. Since emission of each additional photon supresses the decay rate by a factor of order $O(\alpha)$, the dominant electromagnetic decay modes are 2γ (two photons) for the C-even states and 3γ for the C-odd states. (For kinematic reasons, decays into one photon and nothing else are forbidden.)

The so-called *para-positronium* states are spin $S=0$ bound states of an electron and a positron. The ground 1S para-positronium state has $C=(-1)^S(-1)^L=+1$, so its main decay mode produces two photons; the 4γ, 6γ, *etc.*, decay modes are allowed but have very small branching ratios. The $S=1$ positronium states are called *ortho-positronium*; the 1S ortho-positronium state has $C=(-1)^S(-1)^L=-1$, so it usually decays into three photons; the 5γ, *etc.*, decay modes are allowed but rare. Because of an additional final-state photon, the ortho-positronium ground states decays much slower than its para-positronium counterpart.

Problem **3**(a):

Consider the spin-reversal operator $\widehat{\mathcal{SR}}\,|m_s\rangle = i^{2m_s}\,|-m_s\rangle$. For a generic polarization state $|\xi\rangle$ of a non-relativistic spin-half particle, one has $\widehat{\mathcal{SR}}\,|\xi\rangle = \left|\sigma^2\xi^*\right\rangle$, which implies that for a relativistic electron

$$\widehat{\mathcal{SR}}\,u(\mathbf{p},s) \;\equiv\; i^{2m_s}\,u(\mathbf{p},-m_s) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\sigma^2\xi_s^* \\ \sqrt{p\cdot\bar\sigma}\,\sigma^2\xi_s^* \end{pmatrix} \tag{S.13}$$

Let us also reverse the momentum 3-vector, $\mathbf{p}\to-\mathbf{p}$, or in Lorentz notations, $p^\mu\to\tilde p^\mu\equiv(E_{\mathbf{p}},-\mathbf{p})$. Note that $\tilde p\cdot\sigma = p\cdot\bar\sigma$ and $p\cdot\bar\sigma\to p\cdot\sigma$; therefore

$$\begin{aligned} i^{2m_s}\,u(-\mathbf{p},-m_s) \;\equiv\; \widehat{\mathcal{SR}}\,u(-\mathbf{p},s) \;&=\; \begin{pmatrix} \sqrt{p\cdot\bar\sigma}\,\sigma^2\xi_s^* \\ \sqrt{p\cdot\sigma}\,\sigma^2\xi_s^* \end{pmatrix} \;=\; \begin{pmatrix} \sigma^2\sqrt{p\cdot\sigma^*}\,\xi_s^* \\ \sigma^2\sqrt{p\cdot\bar\sigma^*}\,\xi_s^* \end{pmatrix} \\ &=\; \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\,\xi_s \end{pmatrix}^* \;=\; -i\gamma^1\gamma^3\,u^*(\mathbf{p},s), \end{aligned} \tag{S.14}$$

or equivalently $i^{2m_s}u^*(-\mathbf{p},-m_s) = -i\gamma^1\gamma^3u(+\mathbf{p},+m_s)$, which proves the first half of the lemma.

The second half of the lemma can be proved in a similar way, but there is simpler shortcut using $v(\mathbf{p},s)=\gamma^2u^*(\mathbf{p},s)$:

$$\begin{aligned} i^{2m_s}\,v^*(-\mathbf{p},-m_s) \;&=\; \gamma^2\left(i^{2m_s}\,u^*(-\mathbf{p},-m_s)\right)^* \;=\; \gamma^2\left(-i\gamma^1\gamma^3\,u(+\mathbf{p},+m_s)\right)^* \\ &=\; +i\gamma^1\gamma^3\gamma^2\,u^*(+\mathbf{p},+m_s) \;=\; -i\gamma^1\gamma^3\,v(+\mathbf{p},+m_s). \end{aligned} \tag{S.15}$$

Problem **3**(b):

$$\begin{aligned}
\hat{\mathcal{T}}\hat{\Psi}(\mathbf{x},t)\hat{\mathcal{T}}^{-1} &= \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s\Big[\big(\hat{\mathcal{T}}e^{i\mathbf{px}-iEt}u(\mathbf{p},s)\hat{\mathcal{T}}^{-1}\big)\,\big(\hat{\mathcal{T}}\hat{a}(\mathbf{p},s)\hat{\mathcal{T}}^{-1}\big)\\
&\qquad\qquad +\ \big(\hat{\mathcal{T}}e^{-i\mathbf{px}+iEt}v(\mathbf{p},s)\hat{\mathcal{T}}^{-1}\big)\big(\hat{\mathcal{T}}\hat{b}^\dagger(\mathbf{p},s)\hat{\mathcal{T}}^{-1}\big)\Big]\\
&= \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s\Big[\big(e^{-i\mathbf{px}+iEt}u^*(\mathbf{p},s)\big)\,\big((\pm i)i^{2m_s}\,\hat{a}(-\mathbf{p},-s)\big)\\
&\qquad\qquad +\ \big(e^{-i\mathbf{px}+iEt}u^*(\mathbf{p},s)\big)\,\big((\pm i)i^{2m_s}\,\hat{b}^\dagger(-\mathbf{p},-s)\big)\Big]\\
&= \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s\Big[\big(e^{+i\mathbf{px}+iEt}\,(\pm i)i^{2m_s}\,u^*(-\mathbf{p},-s)\big)\,\hat{a}(\mathbf{p},s)\\
&\qquad\qquad +\ \big(e^{-i\mathbf{px}-iEt}\,(\pm i)i^{2m_s}\,v^*(-\mathbf{p},-s)\big)\,\hat{b}^\dagger(\mathbf{p},s)\,\Big]\\
&= \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s\Big[\big(e^{+i\mathbf{px}+iEt}\,(\pm\gamma^1\gamma^3)\,u(\mathbf{p},s)\big)\,\hat{a}(\mathbf{p},s)\\
&\qquad\qquad +\ \big(e^{-i\mathbf{px}-iEt}\,(\pm\gamma^1\gamma^3)\,v(\mathbf{p},s)\big)\,\hat{b}^\dagger(\mathbf{p},s)\,\Big]\\
&= \ \pm\gamma^1\gamma^3\,\hat{\Psi}(\mathbf{x},-t).
\end{aligned}\tag{S.16}$$

$\mathcal{Q.E.D.}$

Problem **3**(c):

$$\begin{aligned}
\hat{\mathcal{T}}\hat{\bar{\Psi}}\Gamma\hat{\Psi}\hat{\mathcal{T}}^{-1}\Big|_{(\mathbf{x},t)} &= \big(\hat{\mathcal{T}}\hat{\Psi}\hat{\mathcal{T}}^{-1}\big)^\dagger\,\big(\hat{\mathcal{T}}\gamma^0\Gamma\hat{\mathcal{T}}^{-1}\big)\,\big(\hat{\mathcal{T}}\hat{\Psi}\hat{\mathcal{T}}^{-1}\big)\Big|_{(\mathbf{x},t)}\\
&= \big(\pm\gamma^1\gamma^3\hat{\Psi}\big)^\dagger\,\big(\gamma^0\Gamma\big)^*\,\big(\pm\gamma^1\gamma^3\hat{\Psi}\big)\Big|_{(\mathbf{x},-t)}\\
&= \hat{\bar{\Psi}}\gamma^3\gamma^1\Gamma^*\gamma^1\gamma^3\hat{\Psi}\Big|_{(\mathbf{x},-t)} \ \equiv\ \hat{\bar{\Psi}}\Gamma^t\hat{\Psi}\Big|_{(\mathbf{x},-t)}.
\end{aligned}\tag{S.17}$$

$\mathcal{Q.E.D.}$

Problem **3**(d):

By inspection, $\mathbf{1}^t = \gamma^3\gamma^1\gamma^1\gamma^3 = +\mathbf{1}$. The γ_5 matrix is real and commutes with the $\gamma^3\gamma^1$, hence $\gamma_5^t = +\gamma^5$. Among the γ_μ matrices, the γ_1 and γ_3 are real but anti-commute with the

$\gamma^3\gamma^1$ while the γ_2 is imaginary but commutes with the $\gamma^3\gamma^1$, hence for all three, $\vec{\gamma}^t = -\vec{\gamma}$. On the other hand, γ_0 is real and commutes with the $\gamma^3\gamma^1$, hence $\gamma_0^t = +\gamma^0$: Time-reversal, like parity, acts differently on space *vs.* time components of all four-vectors, including the γ_μ. Finally, $(\Gamma_1\Gamma_2)^t = \Gamma_1^t\Gamma_2^t$, hence $(\gamma_5\vec{\gamma})^t = -\gamma_5\vec{\gamma}$, $(\gamma_5\gamma_0)^t = +\gamma_5\gamma_0$, $\gamma_{ij}^t = +\gamma_{ij}$ and $\gamma_{i0}^t = -\gamma_{i0}$.

Note however that $\bigl(\hat{\overline{\Psi}}\Gamma\hat{\Psi}\bigr)^\dagger = \hat{\overline{\Psi}}\overline{\Gamma}\hat{\Psi}$ where $\overline{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0$. Hence while the scalar, the vector and the axial vector bilinears are hermitian, the tensor $\hat{\overline{\Psi}}\gamma_{\mu\nu}\hat{\Psi}$ and the pseudoscalar are anti-hermitian — so let us make them hermitian by re-defining

$$\hat{T}_{\mu\nu} \stackrel{\text{def}}{=} \hat{\overline{\Psi}}(i\gamma_{\mu\nu})\hat{\Psi}, \qquad \hat{P} \stackrel{\text{def}}{=} \hat{\overline{\Psi}}(i\gamma_5)\hat{\Psi}. \tag{S.18}$$

Since the time-reversal operator $\hat{\mathcal{T}}$ anti-commutes with the i, the re-definition (S.18) reverses the time-reversal properties of the tensor and pseudoscalar bilinears. Thus

$$\hat{\mathcal{T}}\hat{P}(\mathbf{x},t)\hat{\mathcal{T}}^{-1} = -\hat{P}(\mathbf{x},-t), \quad \hat{\mathcal{T}}\hat{T}_{ij}(\mathbf{x},t)\hat{\mathcal{T}}^{-1} = -\hat{T}_{ij}(\mathbf{x},-t), \quad \hat{\mathcal{T}}\hat{T}_{i0}(\mathbf{x},t)\hat{\mathcal{T}}^{-1} = +\hat{T}_{i0}(\mathbf{x},-t), \tag{S.19}$$

while for the rest of the bilinears

$$\begin{aligned}
\hat{\mathcal{T}}\hat{S}(\mathbf{x},t)\hat{\mathcal{T}}^{-1} &= +\hat{S}(\mathbf{x},-t), \\
\hat{\mathcal{T}}\hat{\mathbf{V}}(\mathbf{x},t)\hat{\mathcal{T}}^{-1} &= -\hat{\mathbf{V}}(\mathbf{x},-t), &\quad \hat{\mathcal{T}}\hat{V}_0(\mathbf{x},t)\hat{\mathcal{T}}^{-1} &= +\hat{V}_0(\mathbf{x},-t), \\
\hat{\mathcal{T}}\hat{\mathbf{A}}(\mathbf{x},t)\hat{\mathcal{T}}^{-1} &= -\hat{\mathbf{A}}(\mathbf{x},-t), &\quad \hat{\mathcal{T}}\hat{A}_0(\mathbf{x},t)\hat{\mathcal{T}}^{-1} &= +\hat{A}_0(\mathbf{x},-t).
\end{aligned} \tag{S.20}$$

<u>Problem **3**(e)</u>:

$$\hat{\mathcal{T}}\hat{\overline{\Psi}}(\mathbf{x},t)\bigl(i\gamma_\mu\partial^\mu - m\bigr)\hat{\Psi}(\mathbf{x},t)\hat{\mathcal{T}}^{-1} = \hat{\overline{\Psi}}(\mathbf{x},-t)\bigl(-i\gamma_\mu^t\partial^\mu - m\bigr)\hat{\Psi}(\mathbf{x},-t) = \hat{\overline{\Psi}}\bigl(i\gamma_\mu\partial^\mu - m\bigr)\hat{\Psi}\Big|_{(\mathbf{x},-t)} \tag{S.21}$$

and hence the Dirac action

$$\mathit{Action} = \int\! d^4x\, \hat{\overline{\Psi}}\bigl(i\gamma_\mu\partial^\mu - m\bigr)\hat{\Psi}$$

is $\mathcal{T}$–invariant.

Problem **4**:

Let us tabulate the $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$ properties of the sixteen hermitian Dirac bilinears:

$\hat{\mathcal{O}}$	$\mathcal{P}$	$\mathcal{C}$	$\mathcal{T}$	$\mathcal{CP}$	$\mathcal{CPT}$
$\hat{S}$	$+$	$+$	$+$	$+$	$+$
$\hat{P}$	$-$	$+$	$-$	$-$	$+$
$\hat{V}^0$	$+$	$-$	$+$	$-$	$-$
$\hat{V}^i$	$-$	$-$	$-$	$+$	$-$
$\hat{A}^0$	$-$	$+$	$+$	$-$	$-$
$\hat{A}^i$	$+$	$+$	$-$	$+$	$-$
$\hat{T}^{ij}$	$+$	$-$	$-$	$-$	$+$
$\hat{T}^{i0}$	$-$	$-$	$+$	$+$	$+$

The last column in this table can be summarized as

$$\widehat{\mathcal{CPT}}\,\hat{\mathcal{O}}(x)\,\widehat{\mathcal{CPT}} \;=\; \hat{\mathcal{O}}(-x)\times(-1)^{\#\text{Lorentz indices in }\hat{\mathcal{O}}}. \tag{S.22}$$

For a non-hermitian bilinear operator $\hat{\mathcal{O}}' = (\text{phase})\hat{\mathcal{O}}$, anti-linearity of the $\widehat{\mathcal{CPT}}$ operator implies $\widehat{\mathcal{CPT}}\hat{\mathcal{O}}'\,\widehat{\mathcal{CPT}} = (\text{phase})^*\,\widehat{\mathcal{CPT}}\,\hat{\mathcal{O}}\,\widehat{\mathcal{CPT}}$. Hence, for any Dirac bilinear, hermitian or not,

$$\widehat{\mathcal{CPT}}\,\hat{\mathcal{O}}(x)\,\widehat{\mathcal{CPT}} \;=\; \hat{\mathcal{O}}^\dagger(-x)\times(-1)^{\#\text{Lorentz indices in }\hat{\mathcal{O}}}. \tag{S.23}$$

$\mathcal{Q.E.D.}$