Textbook Problem **4.2**:

We begin by developing Feynman rules for the theory at hand. The Hamiltonian clearly decomposes into $\hat{H} = \hat{H}_0 + \hat{V}$ where

$$\hat{H}_0 \;=\; \hat{H}_\Phi^{\rm free} \;+\; \hat{H}_\phi^{\rm free} \tag{S.1}$$

and

$$\hat{V} \;=\; \int d^3\mathbf{x}\,\mu\Phi(\mathbf{x})\phi^2(\mathbf{x}). \tag{S.2}$$

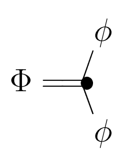
In perturbation theory, Feynman propagators are contractions of the free fields which follow from the free Hamiltonian (S.1). Thus,

$$\begin{aligned} \Phi(x)\bullet\!\!=\!\!=\!\!=\!\!\bullet\,\Phi(y) \;&=\; \overbrace{\Phi(x)\;\Phi}(y) \;=\; D^F(x-y)_{{\rm mass}=M}\,,\\ \phi(x)\bullet\!\!-\!\!-\!\!-\!\!\bullet\,\phi(y) \;&=\; \overbrace{\phi(x)\;\phi}(y) \;=\; D^F(x-y)_{{\rm mass}=m}\,, \end{aligned} \tag{S.3}$$

and there are no mixed $\overbrace{\phi\;\Phi}$ contractions. In the momentum basis, this gives us two distinct propagators,

$$=\!=\!=\!=\!=\!= \;=\; \frac{i}{q^2 - M^2 + i0} \qquad \text{and} \qquad -\!-\!-\!-\!-\!- \;=\; \frac{1}{q^2 - m^2 + i0}\,. \tag{S.4}$$

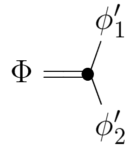
The Feynman vertices follow from the perturbation Hamiltonian (S.2), which involves one power of the Φ field and two powers of the ϕ field, hence the vertices involve one double line and two single lines (net valence = 3),



$$= \; -2i\mu \tag{S.5}$$

where the factor 2 comes from the interchangeability of the two identical $\hat{\phi}$ fields in the vertex.

Now consider the decay process $\Phi \to \phi + \phi$. To the lowest order of the perturbation theory, we have a single diagram



(S.6)

with one vertex, one incoming double line, two outgoing single lines and no internal lines of either kind. Thus,

$$\left\langle \phi'_1 + \phi'_2 \right| i\hat{T} \left| \Phi \right\rangle \equiv \mathcal{M} \times (2\pi)^4 \delta^{(4)}(p - p'_1 - p'_2) = -2i\mu \times (2\pi)^4 \delta^{(4)}(p - p'_1 - p'_2), \quad \text{(S.7)}$$

i.e., $\mathcal{M}(\Phi \to \phi'_1 + \phi'_2) = -2\mu$.

It remains to calculate the decay rate of the Φ particle in its rest frame. Using eq. (4.86) of the textbook, we have

$$\begin{aligned} \Gamma &= \frac{1}{2M} \int \frac{d^3\mathbf{p}'_1}{(2\pi)^3(2E'_1)} \int \frac{d^3\mathbf{p}'_2}{(2\pi)^3(2E'_2)} \, |\mathcal{M}|^2 \times (2\pi)^4 \delta^{(3)}(\mathbf{p}'_1 + \mathbf{p}'_2) \delta(E'_1 + E'_2 - M) \\ &= \frac{1}{2M} \int \frac{d^3\mathbf{p}'_1}{(2\pi)^2(2E'_1)(2E'_2)} \times (4\mu^2)\, \delta(E'_1 + E'_2 - M) \\ &= \frac{\mu^2}{8\pi^2 M} \int d\Omega \int \frac{|\mathbf{p}'_1|^2 d|\mathbf{p}'_1|}{E'_1 E'_2} \, \delta(E'_1 + E'_2 - M) \end{aligned} \quad \text{(S.8)}$$

Now, $\int d\Omega = (4\pi)/2$ because of two identical bosons in the final state, while the rest of the phase-space integral evaluates to

$$\int \frac{|\mathbf{p}'_1|^2 d|\mathbf{p}'_1|}{E'_1 E'_2} \, \delta(E'_1 + E'_2 - M) = \frac{|\mathbf{p}'_1| = \sqrt{(M/2)^2 - m^2}}{M}. \quad \text{(S.9)}$$

Therefore,

$$\Gamma = \frac{\mu^2}{8\pi M} \times \sqrt{1 - \left(\frac{2m}{M}\right)^2}. \quad \text{(S.10)}$$

Textbook Problem **4.3**(a):

Similar to the previous problem, the propagators are contractions of the free fields, thus for N distinct fields Φ^i of the same mass m we have

$$\Phi^j(x) \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet \Phi^k(y) \;=\; \overline{\Phi^j(x)\ \Phi^k(y)} \;=\; \delta^{jk}\, D^F(x-y)_{\text{mass}=m}\,, \tag{S.11}$$

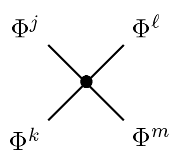
or in momentum space,

$$\Phi^j \;\rule[0.5ex]{6em}{0.4pt}\; \Phi^k \;=\; \frac{i\delta^{jk}}{q^2 - m^2 + i0}\,. \tag{S.12}$$

The vertices follow from the perturbation operator

$$\hat{V} \;=\; \int\! d^3\mathbf{x}\left(\tfrac{\lambda}{4}(\Phi\cdot\Phi)^2 \;\equiv \sum_j \tfrac{\lambda}{4}(\hat{\Phi}^j)^4 \;+ \sum_{j<k} \tfrac{\lambda}{2}(\hat{\Phi}^j)^2(\hat{\Phi}^k)^2\right), \tag{S.13}$$

hence two vertex types: (1) a vertex involving 4 lines of the same field species Φ^j, with the vertex factor $-i\frac{\lambda}{4}\times 4! = -6i\lambda$; and (2) a vertex involving 2 lines of one field species Φ^j and 2 lines of a different species Φ^k, with the vertex factor $-i\frac{\lambda}{2}\times(2!)^2 = -2i\lambda$. (The combinatoric factors arise from the interchanges of the identical fields in the same vertex, thus 4! for the first vertex type and $(2!)^2$ for the second type.) Equivalently, we may use a single vertex type involving 4 fields of whatever species, with the species-dependent vertex factor



$$= \; -2i\lambda\big(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}\big). \tag{S.14}$$

Now consider the scattering process $\Phi^j + \Phi^k \to \Phi^\ell + \Phi^m$. At the lowest order of the perturbation theory, there is just one Feynman diagram for this process; it has one vertex, 4 external legs and no internal lines. Consequently, at the lowest order,

$$\mathcal{M}(\Phi^j + \Phi^k \to \Phi^\ell + \Phi^m) \;=\; -2\lambda\big(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}\big) \tag{S.15}$$

independent of the particles' momenta. Specifically,

$$\begin{aligned}\mathcal{M}(\Phi^1+\Phi^2\to\Phi^1+\Phi^2) &= -2\lambda,\\ \mathcal{M}(\Phi^1+\Phi^1\to\Phi^2+\Phi^2) &= -2\lambda,\\ \mathcal{M}(\Phi^1+\Phi^1\to\Phi^1+\Phi^1) &= -6\lambda,\end{aligned}\tag{S.16}$$

and consequently (using eq. (4.85) of the textbook)

$$\begin{aligned}\frac{d\sigma(\Phi^1+\Phi^2\to\Phi^1+\Phi^2)}{d\Omega_{\rm c.m.}} &= \frac{\lambda^2}{16\pi^2E^2_{\rm c.m.}}\,,\\ \frac{d\sigma(\Phi^1+\Phi^1\to\Phi^2+\Phi^2)}{d\Omega_{\rm c.m.}} &= \frac{\lambda^2}{16\pi^2E^2_{\rm c.m.}}\,,\\ \frac{d\sigma(\Phi^1+\Phi^1\to\Phi^1+\Phi^1)}{d\Omega_{\rm c.m.}} &= \frac{9\lambda^2}{16\pi^2E^2_{\rm c.m.}}\,.\end{aligned}\tag{S.17}$$

These are *partial* cross sections. To calculate the total cross sections, we integrate over $d\Omega$, which gives the factor of 4π when the two final particles are of distinct species, but for the same species, we only get 2π because of Bose statistics. Hence,

$$\begin{aligned}\sigma_{\rm tot}(\Phi^1+\Phi^2\to\Phi^1+\Phi^2) &= \frac{\lambda^2}{4\pi E^2_{\rm c.m.}}\,,\\ \sigma_{\rm tot}(\Phi^1+\Phi^1\to\Phi^2+\Phi^2) &= \frac{\lambda^2}{8\pi E^2_{\rm c.m.}}\,,\\ \sigma_{\rm tot}(\Phi^1+\Phi^1\to\Phi^1+\Phi^1) &= \frac{9\lambda^2}{8\pi E^2_{\rm c.m.}}\,.\end{aligned}\tag{S.18}$$

Textbook Problem **4.3**(b):

The linear sigma model was discussed earlier in class (and on the mid-term exam). For a negative mass term $m^2=-\mu^2<0$, the classical potential

$$V(\Phi^2) \;=\; \tfrac{1}{2}m^2(\Phi^2) \;+\; \tfrac{1}{4}\Lambda(\Phi^2)^2 \tag{S.19}$$

has a minimum (or rather a spherical shell of minima) for

$$\Phi^2\equiv\Phi\cdot\Phi \;=\; v^2 \;=\; \frac{\mu^2}{\lambda} \;>\; 0. \tag{S.20}$$

Semi-classically, we expect a non-zero vacuum expectation value of the scalar fields, $\langle\Phi\rangle\neq 0$

with $\langle\Phi\rangle^2 = v^2$, or equivalently, $\langle\Phi^j\rangle = v\delta^{jN}$ modulo the $O(N)$ symmetry of the problem. Shifting the fields according to

$$\Phi^N(x) \;=\; v \;+\; \sigma(x), \quad \Phi^j(x) \;=\; \pi^j(x)\ (j < N), \tag{S.21}$$

and re-writing the Lagrangian in terms of the shifted fields, we obtain

$$\mathcal{L} \;=\; \tfrac{1}{2}(\partial\sigma)^2 \;-\; \mu^2\sigma^2 \;+\; \tfrac{1}{2}(\partial\boldsymbol{\pi})^2 \;-\; \lambda v\sigma(\sigma^2+\boldsymbol{\pi}^2) \;-\; \tfrac{1}{4}\lambda(\sigma^2+\boldsymbol{\pi}^2)^2 \tag{S.22}$$

(modulo an irrelevant constant term) where $\boldsymbol{\pi}$ stands for the $(N-1)$–plet of the π^j fields, thus $\boldsymbol{\pi}^2 = \sum_j(\pi^j)^2$.

The free part of the Lagrangian (S.22) (the first 3 terms) describe a massive real scalar field $\sigma(x)$ of mass $m_\sigma = \sqrt{2}\mu$ and $(N-1)$ massless real scalars $\pi^j(x)$, which are the Goldstone particles of the $O(N)$ symmetry spontaneously broken down to $O(N-1)$ (thus $(N-1)$ broken symmetry generators, forming a vector multiplet of the unbroken $O(N-1)$ symmetry). Consequently, the non-zero contractions of the free σ and π fields are

$$\begin{aligned} \overline{\sigma(x)\ \sigma}(y) \;&=\; D^F(x-y)_{\text{mass}=m_\sigma}\,, \\ \overline{\pi^j(x)\ \pi}^k(y) \;&=\; \delta^{jk}\,D^F(x-y)_{\text{mass}=0}\,, \end{aligned} \tag{S.23}$$

which give us two distinct Feynman propagators in the momentum basis,

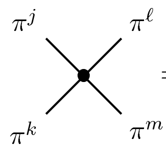
$$\begin{aligned} \sigma \;=\!=\!=\!=\!=\!=\!=\;\sigma \;&=\; \frac{i}{q^2-2\mu^2+i0}\,, \\ \pi^j \;\text{———————}\; \pi^k \;&=\; \frac{i\delta^{jk}}{q^2+i0}\,. \end{aligned} \tag{S.24}$$

The last two terms in the Lagrangian (S.22) give rise to the interaction Hamiltonian of the linear sigma model, namely

$$\hat{V} \;=\; \int\! d^3\mathbf{x}\Big(\lambda v\hat{\sigma}^3 \;+\; \lambda v\hat{\sigma}\hat{\boldsymbol{\pi}}^2 \;+\; \tfrac{\lambda}{4}\hat{\sigma}^4 \;+\; \tfrac{\lambda}{2}\hat{\sigma}^2\hat{\boldsymbol{\pi}}^2 \;+\; \tfrac{\lambda}{4}(\hat{\boldsymbol{\pi}}^2)^2\Big). \tag{S.25}$$

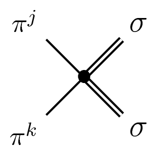
The five terms in this interaction Hamiltonian give rise to five types of Feynman vertices.

Proceeding exactly as in part (a) of the problem, we obtain

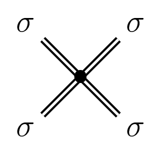


$$= \; -2i\lambda\big(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}\big) \tag{S.26}$$

and similarly

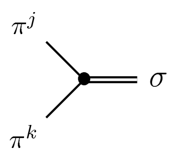


$$= \; -2i\lambda\delta^{jk} \qquad \text{and}$$

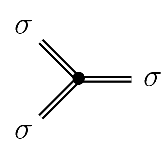


$$= \; -6i\lambda. \tag{S.27}$$

The remaining two vertices have valence = 3 and follow from the cubic terms in the interaction Hamiltonian (S.25). The analysis proceeds exactly as in the previous problem and yields



$$= \; -2i\lambda v\delta^{jk} \qquad \text{and}$$

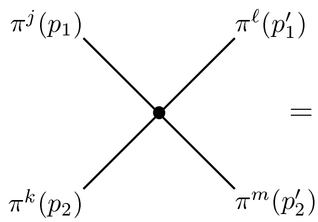


$$= \; -6i\lambda v. \tag{S.28}$$

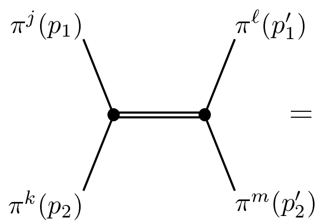
This completes the Feynman rules of the linear sigma model.

Textbook Problem **4.3**(c):

In this part of the problem, we use the Feynman rules we have just derived to calculate the tree-level $\pi\pi \to \pi\pi$ scattering amplitudes. As explained in class, a tree diagram ($L = 0$) with $E = 4$ external legs has either one valence = 4 vertex (and hence no propagators) or two valence = 3 vertices (and hence one propagator). Altogether, there are four such diagrams contributing to the tree-level $i\mathcal{M}\big(\pi^j(p_1) + \pi^k(p_2) \to \pi^\ell(p_1') + \pi^m(p_2')\big)$ — they are shown in the textbook. The diagrams evaluate to:

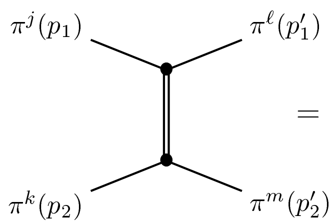


$$= \; -2i\lambda\big(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}\big),$$

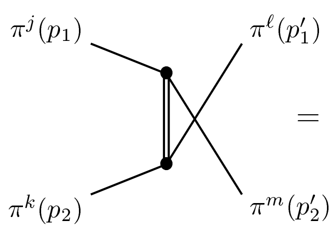


$$= \; (-2i\lambda v\delta^{jk})\,\frac{i}{(p_1+p_2)^2-2\mu^2}\,(-2i\lambda v\delta^{\ell m}),$$

(S.29)



$$= \; (-2i\lambda v\delta^{j\ell})\,\frac{i}{(p_1-p_1')^2-2\mu^2}\,(-2i\lambda v\delta^{km}),$$



$$= \; (-2i\lambda v\delta^{jm})\,\frac{i}{(p_1-p_2')^2-2\mu^2}\,(-2i\lambda v\delta^{k\ell}),$$

which gives the net scattering amplitude

$$\begin{aligned}\mathcal{M}\big(\pi^j(p_1)+\pi^k(p_2)\to\pi^\ell(p_1')+\pi^m(p_2')\big) \;=\; & -2\lambda\delta^{jk}\delta^{\ell m}\left(1+\frac{2\lambda v^2}{(p_1+p_2)^2-2\mu^2}\right)\\ & -2\lambda\delta^{j\ell}\delta^{km}\left(1+\frac{2\lambda v^2}{(p_1-p_1')^2-2\mu^2}\right)\\ & -2\lambda\delta^{jm}\delta^{k\ell}\left(1+\frac{2\lambda v^2}{(p_1-p_2')^2-2\mu^2}\right).\end{aligned} \tag{S.30}$$

Now, according to eq. (S.20), $\lambda v^2=\mu^2$, which makes for

$$\left(1+\frac{2\lambda v^2}{(p_1+p_2)^2-2\mu^2}\right) \;=\; \frac{(p_1+p_2)^2}{(p_1+p_2)^2-2\mu^2} \tag{S.31}$$

and ditto for the other two terms in the amplitude (S.30). Altogether, we now have

$$\mathcal{M} = -2\lambda\Big(\delta^{jk}\delta^{\ell m}\times\frac{(p_1+p_2)^2}{(p_1+p_2)^2-2\mu^2} + \delta^{j\ell}\delta^{km}\times\frac{(p_1-p_1')^2}{(p_1-p_1')^2-2\mu^2} + \delta^{jm}\delta^{k\ell}\times\frac{(p_1-p_2')^2}{(p_1-p_2')^2-2\mu^2}\Big), \tag{S.32}$$

which vanishes in the zero-momentum limit for *any one of the four pions*, initial or final. Indeed, since the pions are massless, $(p_1)^2=(p_2)^2=(p_1')^2=(p_2')^2=0$ and hence

$$\begin{aligned} s &\stackrel{\rm def}{=} (p_1+p_2)^2 \equiv (p_1'+p_2')^2 = 2(p_1p_2) = 2(p_1'p_2'),\\ t &\stackrel{\rm def}{=} (p_1'-p_1)^2 \equiv (p_2'-p_2)^2 = -2(p_1'p_1) = -2(p_2'p_2),\\ u &\stackrel{\rm def}{=} (p_1'-p_2)^2 \equiv (p_1'-p_2)^2 = -2(p_2'p_1) = -2(p_1'p_2), \end{aligned} \tag{S.33}$$

this whenever any one of the four momenta becomes small, all three numerators in the amplitude (S.32) become small as well, thus $\mathcal{M}=O(\text{small}p)$.

Please note that although (S.32) is only the tree-level approximation to the actual scattering amplitude, its behavior in the small pion momentum limit is correct and completely general. According to the *Goldstone theorem*, not only the Goldstone particles (such as 'pions' in this linear sigma model) are exactly massless, but also *any scattering amplitude involving any Goldstone particle vanishes as $O(p_\pi)$ when the Goldstone particle's momentum p_π goes to zero.*

To complete this part of the problem, let us now assume that all four pion's momenta are small compares to the σ-particle's mass $m_\sigma=\sqrt{2}\mu$. In this limit, all three denominators in eq. (S.32) are dominated by the $-2\mu^2$ term, hence

$$\mathcal{M} = \frac{1}{v^2}\left(\delta^{jk}\delta^{\ell m}(p_1+p_2)^2 + \delta^{j\ell}\delta^{km}(p_1-p_1')^2 + \delta^{jm}\delta^{k\ell}(p_1-p_2')^2 + O\left(\frac{p^4}{m_\sigma^2}\right)\right). \tag{S.34}$$

For generic species of the four pions, this amplitude is of the order $O(p^2/v^2)$, but there is a cancellation when all for pions belong to the same species (this is unavoidable for $N=2$).

Indeed,

$$\begin{aligned}(p_1+p_2)^2+(p_1-p_1')^2+(p_1-p_2')^2 \;&=\; 2(p_1p_2)-2(p_1p_1')-2(p_1p_2')\\ &= 2p_1(p_2-p_1'-p_2'=-p_1) \;=\; -2p_1^2=0\end{aligned} \tag{S.35}$$

and hence

$$\mathcal{M}(\pi^1+\pi^1\to\pi^1+\pi^1) \;=\; \frac{1}{v^2}\left(0\;+\;O\left(\frac{p^4}{m_\sigma^2}\right)\right). \tag{S.36}$$

$\mathcal{Q.E.D.}$

Finally, let us translate the amplitude (S.34) into the low-energy scattering cross sections:

$$\begin{aligned}\frac{d\sigma(\pi^1+\pi^2\to\pi^1+\pi^2)}{d\Omega_{\text{c.m.}}} \;&=\; \frac{E^2_{\text{c.m.}}}{64\pi^2v^4}\times\sin^4\frac{\theta_{\text{c.m.}}}{2}\,,\\ \sigma_{\text{tot}}(\pi^1+\pi^2\to\pi^1+\pi^2) \;&=\; \frac{E^2_{\text{c.m.}}}{48\pi v^4}\,,\\ \frac{d\sigma(\pi^1+\pi^1\to\pi^2+\pi^2)}{d\Omega_{\text{c.m.}}} \;&=\; \frac{E^2_{\text{c.m.}}}{64\pi^2v^4}\,,\\ \sigma_{\text{tot}}(\pi^1+\pi^1\to\pi^2+\pi^2) \;&=\; \frac{E^2_{\text{c.m.}}}{32\pi v^4}\,,\\ \sigma(\pi^1+\pi^1\to\pi^1+\pi^1) \;&=\; O\left(\frac{E^6_{\text{c.m.}}}{v^4m_\sigma^4}\right).\end{aligned} \tag{S.37}$$

<u>Textbook Problem **4.3**(d)</u>:
The linear term $\Delta v = -a\Phi^{(N)}$ in the classical potential for the N scalar fields *explicitly* breaks the $O(N)$ symmetry of the theory. Consequently, the potential

$$V(\Phi) \;=\; \tfrac{1}{4}\lambda(\Phi^2)^2 \;-\; \tfrac{1}{2}\mu^2(\Phi^2) \;-\; a\Phi^{(N)} \tag{S.38}$$

now has a *non-degenerate minimum* at

$$\langle\Phi^j\rangle \;=\; v\delta^{jN} \qquad \text{where} \quad v \;\approx\; \sqrt{\frac{\mu}{\lambda}} \;+\; \frac{a}{2\mu} \;+O\left(\frac{a^2\sqrt{\lambda}}{\mu^2\sqrt{\mu}}\right). \tag{S.39}$$

Shifting the fields according to eq. (S.21) for the new value of v now gives us

$$\mathcal{L} \;=\; \tfrac{1}{2}(\partial\sigma)^2 \;-\; \tfrac{1}{2}m_\sigma^2\sigma^2 \;+\; \tfrac{1}{2}(\partial\boldsymbol{\pi})^2 \;-\; \tfrac{1}{2}m_\pi^2\boldsymbol{\pi}^2 -\; \lambda v\sigma(\sigma^2+\boldsymbol{\pi}^2) \;-\; \tfrac{1}{4}\lambda(\sigma^2+\boldsymbol{\pi}^2)^2 \quad \text{(S.40)}$$

(plus an irrelevant constant) where

$$m_\sigma^2 \;=\; 2\mu^2 \;+\; 3m_\pi^2 \qquad \text{and} \qquad m_\pi^2 \;=\; \frac{a}{v} \;>\; 0. \quad \text{(S.41)}$$

Thus, the pions are no longer massive because we no longer have the symmetry whose *spontaneous* breakdown gave rise to massless Goldstone particles. However, for *small* a we have an approximate $O(N)$ symmetry whose small explicit breaking is greatly amplified by the spontaneous breaking. The would-be Goldstone particles of such a symmetry — *e.g.*, pions — are called *pseudo–Goldstone* particles; they have small but non-zero masses proportional to the square root of the symmetry-breaking perturbation such as a. Indeed, the actual π mesons are pseudo–Goldstone particles of the chiral isospin symmetry of the strong interactions. The explicit breaking of this approximate symmetry is due to small masses of the two lightest quarks, hence $m_\pi \propto \sqrt{m_q}$.

Comparing the Lagrangians (S.40) and (S.22) we immediately see identical interaction terms, hence the Feynman vertices of the modified sigma model are exactly as in eqs. (S.26), (S.27) and (S.28), without any modification (except for the new value of v). On the other hand, the Feynman propagators need adjustment to accommodate the new masses (S.41), thus

$$\begin{aligned} \sigma \;{=\!=\!=\!=\!=\!=}\; \sigma \;&=\; \frac{i}{q^2 - m_\sigma^2 + i0}\,, \\ \pi^j \;\text{———}\; \pi^k \;&=\; \frac{i\delta^{jk}}{q^2 - m_\pi^2 + i0}\,. \end{aligned} \quad \text{(S.42)}$$

The tree-level $\pi + \pi \to \pi + \pi$ scattering amplitude is governed by the same four Feynman diagrams as before, thus

$$\begin{aligned} \mathcal{M}\big(\pi^j(p_1) + \pi^k(p_2) \to \pi^\ell(p_1') + \pi^m(p_2')\big) \;=\; &-2\lambda\delta^{jk}\delta^{\ell m}\left(1 \;+\; \frac{2\lambda v^2}{(p_1+p_2)^2 - m_\sigma^2}\right) \\ &- 2\lambda\delta^{j\ell}\delta^{km}\left(1 \;+\; \frac{2\lambda v^2}{(p_1-p_1')^2 - m_\sigma^2}\right) \quad \text{(S.43)} \\ &- 2\lambda\delta^{jm}\delta^{k\ell}\left(1 \;+\; \frac{2\lambda v^2}{(p_1-p_2')^2 - m_\sigma^2}\right), \end{aligned}$$

exactly as in eq. (S.30), except for the new v and new m_σ^2. The exact equation for the minimum (S.39) is

$$\lambda v^2 - \mu^2 = \frac{a}{v} = m_\pi^2 \tag{S.44}$$

hence

$$2\lambda v^2 - m_\sigma^2 = -m_\pi^2 \tag{S.45}$$

and

$$\left(1 + \frac{2\lambda v^2}{(p_1+p_2)^2 - m_\sigma^2}\right) = \frac{(p_1+p_2)^2 - m_\pi^2}{(p_1+p_2)^2 - m_\sigma^2} \tag{S.46}$$

and ditto for the other two terms in the amplitude (S.43). Therefore, instead of eq. (S.32) we now have

$$\begin{aligned}\mathcal{M} = -2\lambda\Big(\delta^{jk}\delta^{\ell m} \times \frac{(p_1+p_2)^2 - m_\pi^2}{(p_1+p_2)^2 - m_\sigma^2} + \delta^{j\ell}\delta^{km} \times \frac{(p_1-p_1')^2 - m_\pi^2}{(p_1-p_1')^2 - m_\sigma^2} \\ + \delta^{jm}\delta^{k\ell} \times \frac{(p_1-p_2')^2 - m_\pi^2}{(p_1-p_2')^2 - m_\sigma^2}\Big),\end{aligned} \tag{S.47}$$

which in the low-energy limit $E_{\text{c.m.}} \ll m_\sigma$ simplifies to

$$\begin{aligned}\mathcal{M} = \left(\frac{2\lambda}{m_\sigma^2} \approx \frac{1}{v^2}\right)\Big(\delta^{jk}\delta^{\ell m}\left((p_1+p_2)^2 - m_\pi^2\right) + \delta^{j\ell}\delta^{km}\left((p_1-p_1')^2 - m_\pi^2\right) \\ + \delta^{jm}\delta^{k\ell}\left((p_1-p_2')^2 - m_\pi^2\right) + O\Big(\frac{p^4}{m_\sigma^2}\Big)\Big).\end{aligned} \tag{S.48}$$

In particular, near the threshold $(p_1+p_2)^2 = E_{\text{c.m.}}^2 \approx 4m_\pi^2$ while $(p_1'-p_1)^2 \approx (p_2'-p_1)^2 \approx 0$ and hence

$$\mathcal{M} \approx \frac{m_\pi^2}{v^2} \times \left(3\delta^{jk}\delta^{\ell m} - \delta^{j\ell}\delta^{km} - \delta^{jm}\delta^{k\ell}\right). \tag{S.49}$$

This threshold amplitude does not vanish. Instead,

$$\mathcal{M} \sim \frac{m_\pi^2}{v^2} = \frac{a}{v^3}. \tag{S.50}$$

Q.E.D.