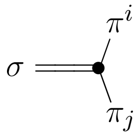
Problem **1**:

At the tree level, the $\sigma \to \pi\pi$ decay proceeds via the Feynman diagram

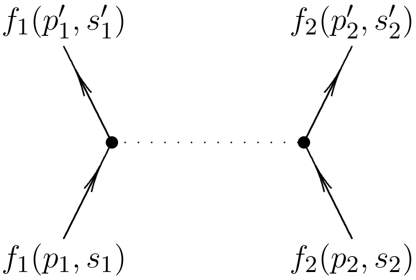


which gives $i\mathcal{M}(\sigma \to \pi^i + \pi^j) = -2i\lambda v\delta^{ij}$. The two pions must have same 'flavor' index $i = j = 1, 2, \ldots, (N-1)$, but the amplitude — and hence decay rate — is the same for all flavors. Thus,

$$\begin{aligned}
\Gamma(\sigma \to \text{any}\,\pi\pi) &= (N-1)\Gamma(\sigma \to \pi^1\pi^1) \\
&= \frac{(N-1)}{2M_\sigma} \iint \frac{d^3\mathbf{p}_1\, d^3\mathbf{p}_2}{(2\pi)^6\, 2E_1\, 2E_2}\, (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_\sigma) \times |-2\lambda v|^2 \\
&= \frac{4(N-1)\lambda^2 v^2}{2M_\sigma} \int \frac{4\pi|\mathbf{p}|^2 d|\mathbf{p}|}{(2\pi)^2 (2E)^2}\, \delta(2E - M_\sigma)\Big|_{E=E_\mathbf{p}=|\mathbf{p}|} \\
&= \frac{(N-1)\lambda^2 v^2}{4\pi M_\sigma} = \frac{(N-1)\lambda}{8\pi}\, M_\sigma\,.
\end{aligned} \tag{S.1}$$

Problem **2**:

As discussed in class, scattering of fermions in the Yukawa theory proceeds by exchange of virtual scalar quanta between the fermions. For the problem at hand, the two fermions involved are of distinct types, so there is just one tree-level Feynman diagram,



which results in the scattering amplitude

$$\mathcal{M}(f_1 + f_2 \to f_1 + f_2) = -\frac{g_1 g_2}{q^2 - M_s^2}\, \bar{u}(p_1', s_1') u(p_1, s_1)\, \bar{u}(p_2', s_2') u(p_2, s_2) \tag{S.2}$$

where $q = p_1' - p_1 = p_2 - p_2'$.

The un-polarized 2-particle scattering cross-section is given by

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} &= \frac{1}{64\pi^2E^2_{\text{c.m.}}}\times\tfrac{1}{2}\sum_{s_1}\tfrac{1}{2}\sum_{s_2}\sum_{s'_1}\sum_{s'_2}|\mathcal{M}|^2\\ &= \frac{1}{64\pi^2E^2_{\text{c.m.}}}\times\left(\frac{g_1g_2}{q^2-M_s^2}\right)\times\tfrac{1}{2}\sum_{s_1,s'_1}\left|\bar{u}(p'_1,s'_1)u(p_1,s_1)\right|^2\times\tfrac{1}{2}\sum_{s_2,s'_2}\left|\bar{u}(p'_2,s'_2)u(p_2,s_2)\right|^2\end{aligned}\tag{S.3}$$

where the spin sums on the second line evaluate according to eq. (2):

$$\begin{aligned}\tfrac{1}{2}\sum_{s_1,s'_1}\left|\bar{u}(p'_1,s'_1)u(p_1,s_1)\right|^2 &= 2(m_1^2+p_1p'_1)\\ &= 2(m_1^2+E_1^2-\mathbf{p}_1^2\cos\theta) \;=\; 4m_1^2\;+\;2\mathbf{p}_1^2(1-\cos\theta)\\ &\quad\langle\!\langle\text{in the center-of-mass frame}\rangle\!\rangle,\\ \tfrac{1}{2}\sum_{s_2,s'_2}\left|\bar{u}(p'_2,s'_2)u(p_2,s_2)\right|^2 &= 2(m_2^2+p_2p'_2)\\ &= 2(m_2^2+E_2^2-\mathbf{p}_2^2\cos\theta) \;=\; 4m_2^2\;+\;2\mathbf{p}_2^2(1-\cos\theta)\\ &\quad\langle\!\langle\text{in the center-of-mass frame}\rangle\!\rangle.\end{aligned}\tag{S.4}$$

Also, in the center of mass frame, $|\mathbf{p}_1|=|\mathbf{p}_2|$, $q^0=0$ and $\mathbf{q}^2=2(1-\cos\theta)\mathbf{p}^2$. Substituting all these formulae into eq. (S.3), we arrive at the partial cross section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} = \frac{g_1^2g_2^2}{64\pi^2E^2_{\text{c.m.}}}\times\frac{(4m_1^2+\mathbf{q}^2)(4m_2^2+\mathbf{q}^2)}{(M_s^2+\mathbf{q}^2)^2}\,.\tag{S.5}$$

Next, we calculate the total cross-section by integrating over scattered particles' directions. To that end, we notice that in the center-of-mass frame,

$$d\Omega \;=\; 2\pi\,d(-\cos\theta) \;=\; \frac{2\pi}{2\mathbf{p}^2}\,d\mathbf{q}^2\tag{S.6}$$

and therefore,

$$\begin{aligned}\sigma_{\text{tot}} &= \frac{g_1^2g_2^2}{64\pi^2E^2_{\text{c.m.}}}\times\frac{2\pi}{2\mathbf{p}^2}\int\limits_0^{4\mathbf{p}^2}d\mathbf{q}^2\,\frac{(4m_1^2+\mathbf{q}^2)(4m_2^2+\mathbf{q}^2)}{(M_s^2+\mathbf{q}^2)^2}\\ &= \frac{g_1^2g_2^2}{16\pi E^2_{\text{c.m.}}}\left[1\;+\;\frac{(4m_1^2-M_s^2)(4m_2^2-M_s^2)}{M_s^2(M_s^2+4\mathbf{p}^2)}\;+\;\frac{2m_1^2+2m_2^2-M_s^2}{2\mathbf{p}^2}\log\frac{M_s^2+4\mathbf{p}^2}{M_s^2}\right]\end{aligned}\tag{S.7}$$

where

$$\mathbf{p}^2 \;=\; \tfrac{1}{4}E_{\rm c.m.}^2 \;-\; \tfrac{1}{2}(m_1^2+m_2^2) \;+\; \frac{(m_1^2-m_2^2)^2}{4E_{\rm c.m.}^2} \tag{S.8}$$

is the solution of the kinematical relation

$$E_{\rm c.m.} \;=\; E_1 \;+\; E_2 \;=\; \sqrt{m_1^2+\mathbf{p}^2} \;+\; \sqrt{m_2^2+\mathbf{p}^2}.$$

It remains to prove eq. (2) we have used above to derive eqs. (S.4). We begin by evaluating a simpler spin sum, for an arbitrary constant spinor w:

$$\begin{aligned}\sum_s |\overline{w}u(p,s)|^2 \;&=\sum_s\Bigl(\overline{w}u(p,s)\;\equiv\sum_\alpha \overline{w}_\alpha u_\alpha(p,s)\Bigr)\Bigl(\bigl(\overline{w}u(p,s)\bigr)^* \;=\; \bar u(p,s)w\;\equiv\sum_\beta \bar u_\beta(p,s)w_\beta\Bigr)\\ &=\sum_{\alpha,\beta}\overline{w}_\alpha\,\Bigl(\sum_s u_\alpha(p,s)\bar u_\beta(p,s)\;=\;(\not p+m)_{\alpha\beta}\Bigr)\,w_\beta\\ &\equiv\;\overline{w}(\not p+m)w.\end{aligned} \tag{S.9}$$

Next, we substitute $w=u(p',s')$ and sum over the spin s':

$$\begin{aligned}\sum_{s,s'}\bigl|\bar u(p',s')u(p,s)\bigr|^2 \;&=\sum_{s'}\Bigl(\bar u(p',s')(\not p+m)u(p',s')\;\equiv\sum_{\alpha,\beta}\bar u_\alpha(p',s')(\not p+m)_{\alpha\beta}u_\beta(p',s')\Bigr)\\ &=\sum_{\alpha,\beta}(\not p+m)_{\alpha\beta}\times\Bigl(\sum_{s'}u_\beta(p',s')\bar u_\alpha(p',s')\;=\;(\not p'+m)_{\beta\alpha}\Bigr)\\ &=\;\operatorname{tr}\bigl((\not p+m)(\not p'+m)\bigr).\end{aligned} \tag{S.10}$$

This proves the first equality in eq. (2); to prove the second equality, we need to evaluate the trace. There is a whole ‘technology’ for evaluating various Dirac traces and we shall study it in January, but the trace we need is simple enough to calculate by inspection of explicit Dirac matrices. In the Weyl basis,

$$(\not p+m)\;=\;\begin{pmatrix} m & p^\mu\sigma_\mu\\ p^\mu\bar\sigma_\mu & m\end{pmatrix},\qquad (\not p'+m)\;=\;\begin{pmatrix} m & p'^\nu\sigma_\nu\\ p'^\nu\bar\sigma_\nu & m\end{pmatrix}, \tag{S.11}$$

hence

$$\begin{aligned}\operatorname{tr}\bigl((\not p+m)(\not p'+m)\bigr)\;&=\;\operatorname{tr}\begin{pmatrix} m^2+p^\mu p'^\nu\sigma_\mu\bar\sigma_\nu & ***\\ *** & m^2+p^\mu p'^\nu\bar\sigma_\mu\sigma_\nu\end{pmatrix}\\ &=\;2m^2\operatorname{tr}(1_{2\times2})\;+\;p^\mu p'^\nu\operatorname{tr}(\sigma_\mu\bar\sigma_\nu+\bar\sigma_\mu\sigma_\nu).\end{aligned} \tag{S.12}$$

For the σ matrices we have $\sigma_0=\bar\sigma_0=1$ while $\sigma_i=-\bar\sigma_i$ are Pauli matrices, which are traceless and satisfy $\operatorname{tr}(\sigma_i\sigma_j) = 2\delta_{ij}$. Consequently,

$$\operatorname{tr}(\sigma_\mu\bar\sigma_\nu) \;=\; \operatorname{tr}(\bar\sigma_\mu\sigma_\nu) \;=\; 2g_{\mu\nu} \tag{S.13}$$

and hence the last line in eq. (S.12) evaluates to $4m^2 + 4pp'$. In other words,

$$\operatorname{tr}\big((\not{p}+m)(\not{p}'+m)\big) \;=\; 4(m^2+pp') \;=\; 4(m^2+EE'-\mathbf{p}\cdot\mathbf{p}'), \tag{S.14}$$

which proves the second equality in eq. (2), $\mathcal{Q.E.D.}$

Problem **3**(a):

To lowest order in $\hat H_I$,

$$\langle\text{out}|\,i\hat T\,|\text{in}\rangle \;=\; \langle\text{out}|\;\text{T-exp}\Big(-i\!\int\! dt\,\hat H_I(t)\Big) - 1\,|\text{in}\rangle \;\approx\; -i\!\int\! dt\;\langle\text{out}|\,\hat H_I(t)\,|\text{in}\rangle\,, \tag{S.15}$$

which for the problem at hand means

$$\begin{aligned}\big\langle e^-(p',s')\big|\,i\hat T\,\big|e^-(p,s)\big\rangle \;&\approx\; ie\!\int\! d^4x\,A_\mu(x)\,\big(e^{+ip'x}\bar u(p',s')\big)\gamma^\mu\big(e^{-ipx}u(p,s)\big)\\ &=\; ie\,\bar u(p',s')\gamma^\mu u(p,s)\times\left[\int\! d^4x\,A_\mu(x)\,e^{ip'x-ipx}\;\equiv\tilde A_\mu(p'-p)\right].\end{aligned} \tag{S.16}$$

Note opposite signs between this formula and its analogue in the textbook. The difference is due to different sign conventions for the e: The textbook uses $e<0$ while I (and most other people) use $e>0$.

Problem **3**(b):

Strictly speaking, the electron scatters off the potential's source, which is basically a heavy particle — *e.g.*, an atomic nucleus — or a system of particles. The static-source approximation arises when the source S is so heavy that its velocity — in some frame — is negligible both before and after the scattering event and its recoil un-observable. In this static-source frame, $E_S = E'_S = M_S$ regardless of the $\mathbf{p}_S$ and $\mathbf{p}'_S$, thus conservation of the total energy of the electron plus the source implies the electron's energy conservation, $E'_e = E_e$.

Now consider eq. (4.79) of the textbook for the scattering cross-section; in the static-source frame, we have

$$\begin{aligned} d\sigma &= \frac{1}{(2E_e)(2M_S)|\mathbf{v}_e|} \times \frac{d^3\mathbf{p}'_e}{(2\pi)^3(2E'_e)} \times \frac{d^3\mathbf{p}'_S}{(2\pi)^3(2M_S)} \times |\mathcal{M}|^2 \times (2\pi)^4\delta^{(4)}(p'_e + p'_S - p_e - p_S) \\ &= \frac{1}{(2E_e)|\mathbf{v}_e|} \times \frac{d^3\mathbf{p}'_e}{(2\pi)^3(2E'_e)} \times \frac{|\mathcal{M}(e+S\to e'+S')|^2}{(2M_S)^2} \times (2\pi)\delta(E'_e - E_e). \end{aligned} \tag{S.17}$$

Note that the amplitude $\mathcal{M}(e+S\to e'+S')$ here is normalized to

$$\left\langle e'+S'\right|\hat{T}\left|e+S\right\rangle = \mathcal{M}(e+S\to e'+S') \times (2\pi)^4\delta^{(4)}(p'_e + p'_S - p_e - p_S), \tag{S.18}$$

but in terms of an electron scattering off a static potential rather than S it is more convenient to use $\mathcal{M}(e\to e')$ normalized to

$$\left\langle e'\right|\hat{T}\left|e\right\rangle = \mathcal{M}(e\to e') \times (2\pi)\delta(E'_e - E_e) \tag{S.19}$$

(this is the normalization used in the problem). The relation between the two amplitudes follows from the relativistic normalization of the source particle's states,

$$\left\langle S'|S\right\rangle = (2E = 2M_S)\,(2\pi)^3\delta^{(3)}(\mathbf{p}'_S - \mathbf{p}_S). \tag{S.20}$$

Putting eqs. (S.18), (S.19) and (S.20) together, we obtain

$$\mathcal{M}(e+S\to e'+S') = (2E = 2M_S) \times \mathcal{M}(e\to e') \tag{S.21}$$

and hence

$$d\sigma = \frac{1}{(2E_e)|\mathbf{v}_e|} \times \frac{d^3\mathbf{p}'_e}{(2\pi)^3(2E'_e)} \times \left|\mathcal{M}(e\to e')\right|^2 \times (2\pi)\delta(E'_e - E_e). \tag{S.22}$$

$\mathcal{Q.E.D.}$

For the purpose of an actual calculation, we integrate eq. (S.22) over the magnitude $|\mathbf{p}'_e|$ of the final electron. This removes the remaining δ–function and simplifies the rest of the kinematic factors, the net result being

$$\left(\frac{d\sigma}{d\Omega_e}\right) = \frac{1}{16\pi^2}\times\left|\mathcal{M}(e\to e')\right|^2. \tag{S.23}$$

Problem **3**(c):

For the Coulomb source, we have $\tilde{\mathbf{A}}(\mathbf{q})=0$, $\tilde{A}_0(\mathbf{q})=Ze/\mathbf{q}^2$ and hence

$$\mathcal{M}\big(e(p,s)\to e(p',s')\big) = \frac{-Ze^2}{\mathbf{q}^2}\times\bar{u}(p',s')\gamma^0 u(p,s). \tag{S.24}$$

For non-relativistic electrons, $\bar{u}(p',s')\gamma^0u(p,s)\approx 2m_e\xi'^\dagger\xi$, the scattering is spin-preserving and spin-independent, and

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{m_e^2}{4\pi^2}\left(\frac{Ze^2}{\mathbf{q}^2}\right)^2 = \frac{\alpha^2Z^2}{4m_e^2v_e^4\,\sin^4(\theta/2)} \tag{S.25}$$

where the second equality follows from $\mathbf{q}^2=2\mathbf{p}_e^2(1-\cos\theta)=(2m_ev_e\sin(\theta/2))^2$.

Problem **3**(d):

For the relativistic electrons, the Coulomb scattering is no longer spin-blind. For an un-polarized stream of initial electrons and a detector blind to the final electrons' spins, we should sum $|\mathcal{M}(e(p,s)\to e(p',s'))|^2$ over the final spin states s' and average over the initial spin states s. Thus,

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega_e}\right) &= \frac{1}{32\pi^2}\sum_{s,s'}\left|\mathcal{M}(e(p,s)\to e(p',s'))\right|^2\\ &= \frac{1}{32\pi^2}\left(\frac{Ze^2}{\mathbf{q}^2}\right)^2\sum_{s,s'}\left|\bar{u}(p',s')\gamma^0u(p,s)\right|^2.\end{aligned} \tag{S.26}$$

According to eq. (3),

$$\sum_{s,s'}\left|\bar{u}(p',s')\gamma^0u(p,s)\right|^2 = 4(m_e^2+E_eE_e'+\mathbf{p}_e\mathbf{p}_e') = 8m_e^2 + 4\mathbf{p}_e^2(1+\cos\theta), \tag{S.27}$$

which gives us the *Mott's formula* for relativistic Coulomb scattering,

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{m_e^2+\mathbf{p}_e^2\cos^2(\theta/2)}{4\pi^2}\left(\frac{Ze^2}{\mathbf{q}^2}\right)^2 = \frac{\alpha^2Z^2}{4m_e^2v_e^4\,\sin^4(\theta/2)}\times\frac{1-\beta_e^2\sin^2(\theta/2)}{\gamma_e^2} \tag{S.28}$$

where $\beta_e\equiv v_e/c$ and $\gamma_e\equiv 1/\sqrt{1-\beta_e^2}$.

It remains to prove eq. (3) for the spin sum. Proceeding exactly as in eqs. (S.9) and (S.10) of problem 2, we derive

$$\sum_{s,s'} \left|\bar{u}(p',s')\gamma^0 u(p,s)\right|^2 \;=\; \operatorname{tr}\left((\not{p}'+m)\gamma^0(\not{p}+m)\gamma^0\right). \tag{3.1}$$

Next, we observe that

$$\gamma^0(\not{p}+m)\gamma^0 \;=\; \gamma^0(E\gamma^0-\mathbf{p}\cdot\vec{\gamma}-m)\gamma^0 = \; E\gamma^0 \;+\; \mathbf{p}\cdot\vec{\gamma} \;+\; m \;=\not{\tilde{p}} \;+\; m \tag{S.29}$$

where $\tilde{p}^\mu = (+E, -\mathbf{p})$. Consequently,

$$\begin{aligned} \operatorname{tr}\left((\not{p}'+m)\gamma^0(\not{p}+m)\gamma^0\right) \;&=\; \operatorname{tr}\left((\not{p}'+m)(\not{\tilde{p}}+m)\right) \\ &\langle\!\langle \text{using eq. (S.14)} \rangle\!\rangle \\ &=\; 4m^2 \;+\; 4p'\tilde{p} \\ &=\; 4m^2 \;+\; 4E'E \;+\; 4\mathbf{p}'\mathbf{p}. \end{aligned} \tag{3.2}$$

$\mathcal{Q}.\mathcal{E}.\mathcal{D}.$