Problem **1**:

Note the correct muon decay amplitude

$$\mathcal{M}(\mu^- \to e^- \nu_\mu \bar{\nu}_e) \;=\; \frac{G_F}{\sqrt{2}} \left[\bar{u}(\nu_\mu)(1-\gamma^5)\gamma^\alpha u(\mu^-)\right] \times \left[\bar{u}(e^-)(1-\gamma^5)\gamma_\alpha v(\bar{\nu}_e)\right]. \tag{1}$$

The complex conjugate of this amplitude

$$\mathcal{M}^* \;=\; \frac{G_F}{\sqrt{2}} \left[\bar{u}(\mu^-)\gamma^\beta(1+\gamma^5)u(\nu_\mu)\right] \times \left[\bar{v}(\bar{\nu}_e)\gamma_\beta(1+\gamma^5)u(e^-)\right] \tag{S.1}$$

has the opposite sign of the γ^5 because $\bar{\gamma}^5 \equiv \gamma^0(\gamma^5)^\dagger\gamma^0 = -\gamma^5$. Consequently,

$$\begin{aligned}|\mathcal{M}|^2 \;=\; \tfrac{1}{2}G_F^2 &\left[\bar{u}(\nu_\mu)(1-\gamma^5)\gamma^\alpha u(\mu^-)\bar{u}(\mu^-)\gamma^\beta(1+\gamma^5)u(\nu_\mu)\right] \\ &\times \left[\bar{u}(e^-)(1-\gamma^5)\gamma_\alpha v(\bar{\nu}_e)\bar{v}(\bar{\nu}_e)\gamma_\beta(1+\gamma^5)u(e^-)\right]\end{aligned} \tag{S.2}$$

and hence

$$\begin{aligned}\tfrac{1}{2}\sum_{\substack{\text{all}\\ \text{spins}}} |\mathcal{M}|^2 \;&=\; \tfrac{1}{4}G_F^2 \operatorname{tr}\Big((1-\gamma^5)\gamma^\alpha(\not{p}_\mu + M_\mu)\gamma^\beta(1+\gamma^5)(\not{p}_{\nu_\mu} + m_{\nu_\mu})\Big) \\ &\qquad \times \operatorname{tr}\Big((1-\gamma^5)\gamma_\alpha(\not{p}_{\bar{\nu}_e} - m_{\nu_e})\gamma_\beta(1+\gamma^5)(\not{p}_e + m_e)\Big) \\ &\approx\; \tfrac{1}{4}G_F^2 \operatorname{tr}\Big((1-\gamma^5)\gamma^\alpha(\not{p}_\mu + M_\mu)\gamma^\beta(1+\gamma^5)\,\not{p}_\nu\Big) \\ &\qquad \times \operatorname{tr}\Big((1-\gamma^5)\gamma_\alpha\,\not{p}_{\bar{\nu}}\gamma_\beta(1+\gamma^5)\,\not{p}_e\Big)\end{aligned} \tag{S.3}$$

where the approximation is $m_e \approx 0$; we also make use of $m_{\nu_e} = m_{\nu_\mu} = 0$ (which may be exactly true or just a very good approximation, future data will tell) and simplify notations: $p_\nu \equiv p_{\nu_\mu}$ and $p_{\bar{\nu}} \equiv p_{\bar{\nu}_e}$. Please note that here and henceforth the indices $\mu, \nu, \bar{\nu}, e$ denote the particles to which respective momenta belong and have nothing to do with the Lorentz indices of those momenta. For the Lorentz indices, I use here α, β and later also γ, δ, σ and ρ. Thus, $p_{\mu\alpha}$ is the α's component of the muon's 4–momentum, *etc.*, *etc.*

Having derived eq. (S.3), we now need to evaluate the traces. For the first trace, we have

$$\begin{aligned}
\operatorname{tr}\Big((1-\gamma^5)\gamma^\alpha(\not{p}_\mu+M_\mu)\gamma^\beta(1+\gamma^5)\,\not{p}_\nu\Big) &= \operatorname{tr}\Big((1-\gamma^5)\gamma^\alpha(\not{p}_\mu+M_\mu)\gamma^\beta\,\not{p}_\nu(1-\gamma^5)\Big)\\
&= \operatorname{tr}\Big((1-\gamma^5)^2\gamma^\alpha(\not{p}_\mu+M_\mu)\gamma^\beta\,\not{p}_\nu\Big)\\
&= 2\operatorname{tr}\Big((1-\gamma^5)\gamma^\alpha(\not{p}_\mu+M_\mu)\gamma^\beta\,\not{p}_\nu\Big)\\
\langle\!\langle\text{using }\operatorname{tr}(\gamma^\alpha\gamma^\beta\,\not{p}_\nu)=\operatorname{tr}(\gamma^5\gamma^\alpha\gamma^\beta\,\not{p}_\nu)=0\rangle\!\rangle &\\
&= 2\operatorname{tr}\Big(\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{p}_\nu\Big)\;-\;2\operatorname{tr}\Big(\gamma^5\gamma^\alpha\,\not{p}_\mu\gamma^\beta\,\not{p}_\nu\Big)\\
&= 8\Big[p_\mu^\alpha p_\nu^\beta\;+\;p_\mu^\beta p_\nu^\alpha\;-\;g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big]\;+\;8i\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}.
\end{aligned}\tag{S.4}$$

Similarly, the second trace evaluates to

$$\operatorname{tr}\big((1-\gamma^5)\gamma_\alpha\,\not{p}_e\gamma_\beta(1+\gamma^5)\,\not{p}_{\bar\nu}\big)\;=\;8\left[(p_{e\alpha}p_{\bar\nu\beta}\;+\;p_{e\beta}p_{\bar\nu\alpha}\;-\;g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\right]\;+\;8i\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\,. \tag{S.5}$$

It remains to substitute the trace formulæ (S.4) and (S.5) back into eq. (S.3) and contract the Lorentz indices. Thus,

$$\begin{aligned}
\tfrac12\sum_{\substack{\text{all}\\\text{spins}}}|\mathcal{M}|^2 \;&=\; 16G_F^2\bigg(\Big[p_\mu^\alpha p_\nu^\beta\;+\;p_\mu^\beta p_\nu^\alpha\;-\;g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big]\;+\;i\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}\bigg)\\
&\qquad\times\bigg(\Big[p_{e\alpha}p_{\bar\nu\beta}\;+\;p_{e\beta}p_{\bar\nu\alpha}\;-\;g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\Big]\;+\;i\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\bigg)\\
&\langle\!\langle\text{using symmetry/antisymmetry of factors under }\alpha\leftrightarrow\beta\,\rangle\!\rangle\\
&=\;16G_F^2\bigg(\Big[p_\mu^\alpha p_\nu^\beta\;+\;p_\mu^\beta p_\nu^\alpha\;-\;g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big]\times\Big[p_{e\alpha}p_{\bar\nu\beta}\;+\;p_{e\beta}p_{\bar\nu\alpha}\;-\;g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\Big]\\
&\qquad\qquad-\;\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}\times\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\bigg)\\
&=\;16G_F^2\bigg(\Big[2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu})\;+\;2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e)\\
&\qquad\qquad-\;2(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu})\;-\;2(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu})\;+\;4(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu})\Big]\\
&\qquad+\;2\Big[2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e)\;-\;2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu})\Big]\bigg)\\
&=\;64G_F^2\,(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e)\,.
\end{aligned}\tag{S.6}$$

$\mathcal{Q.E.D.}$

Lemma:

Consider a decay or a scattering process producing three final state particles. For any such process, the phase space of the final states is

$$d\mathcal{P} \;=\; \frac{d^3\mathbf{p}_1}{(2\pi)^3(2E_1)}\,\frac{d^3\mathbf{p}_2}{(2\pi)^3(2E_2)}\,\frac{d^3\mathbf{p}_3}{(2\pi)^3(2E_3)}\,(2\pi)^3\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\,(2\pi)\delta(E_1+E_2+E_3-E_{\mathrm{tot}}) \tag{S.7}$$

in the center-of-mass frame. In this lemma, we shall see that regardless of the three particles' masses, the phase space (S.7) can be written as

$$d\mathcal{P} \;=\; \frac{d^3\Omega}{256\pi^5}\times dE_1\,dE_2\,dE_3\,\delta(E_1+E_2+E_3-E_{\mathrm{tot}}) \tag{S.8}$$

where $d^3\Omega$ refers to three angular variables parametrizing the directions of the three particles relative to some external frame but not affecting the angles between the three momenta. For example, one may use two angles to describe the orientation of the decay plane (the three momenta are coplanar, $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0$) and one more angle to fix the direction of *e.g.*, $\mathbf{p}_1$ in tha plane. Altogether, $\int d^3\Omega = 4\pi\times 2\pi = 8\pi^2$.

We begin the proof with the obvious step of eliminating the momentum conservation delta-function, thus

$$d\mathcal{P} \;=\; \frac{d^3\mathbf{p}_1\,d^3\mathbf{p}_2}{256\pi^5}\,\frac{\delta(E_1+E_2+E_3-E_{\mathrm{tot}})}{E_1E_2E_3}\bigg|_{\mathbf{p}_3=-(\mathbf{p}_1+\mathbf{p}_2)}. \tag{S.9}$$

Next, we use spherical coordinates for the two remaining momenta,

$$d^3\mathbf{p}_1 \;=\; p_1^2\,dp_1\,d^2\Omega_1\,, \qquad d^3\mathbf{p}_2 \;=\; p_2^2\,dp_2\,d^2\Omega_2\,, \tag{S.10}$$

and then replace the $d^2\Omega_2$ describing the direction of the second particle's momentum relative to the fixed external frame with

$$d^2\Omega_2^{(1)} \;=\; d\theta_{12}\,\sin\theta_{12}\,d\phi_2^{(1)}$$

describing the same direction of $\mathbf{p}_2$ relative to the frame centered on the $\mathbf{p}_1$. Consequently,

$$d^2\Omega_1\,d^2\Omega_2 \;=\; d^2\Omega_1\,d^2\Omega_2^{(1)} \;=\; \left[d^2\Omega_1\,d\phi_2^{(1)}\right]\,d\theta_{12}\,\sin\theta_{12} \;\equiv\; d^3\Omega\times d(\cos\theta_{12}) \tag{S.11}$$

and hence

$$d\mathcal{P} \;=\; \frac{d^3\Omega}{256\pi^5}\times\frac{p_1^2p_2^2}{E_1E_2E_3}\,dp_1\,dp_2\,d(\cos\theta_{12})\,\delta(E_1+E_2+E_3-E_{\rm tot})\Big|_{\mathbf{p}_3=-(\mathbf{p}_1+\mathbf{p}_2)}\,. \tag{S.12}$$

Next, we use the cosine theorem

$$p_3^2 \;=\; (\mathbf{p}_1+\mathbf{p}_2)^2 \;=\; p_1^2\,+\,p_2^2\,+\,2p_1p_2\cos\theta_{12}$$

which gives

$$d(\cos\theta_{12}) \;=\; \frac{p_3\,dp_3}{p_1\,p_2}$$

(for fixed p_1, p_2) and therefore

$$d\mathcal{P} \;=\; \frac{d^3\Omega}{256\pi^5}\times\frac{p_1p_2p_3}{E_1E_2E_3}\,dp_1\,dp_2\,dp_3\,\delta(E_1+E_2+E_3-E_{\rm tot}). \tag{S.13}$$

Finally, we notice that for a relativistic particle of any mass, $pdp = EdE$ and hence (S.8).

It remains to determine the limits of kinematically allowed ways to distribute the net energy $E_{\rm tot}$ of the process among the three final particles. Such limits follow from the triangle inequalities for the three momenta,

$$p_1 \;\le\; p_2\;+\;p_3\,,\qquad p_2\;\le\;p_1\;+\;p_3\,,\qquad p_3\;\le\;p_1\;+\;p_1\,, \tag{S.14}$$

which look simple but produce rather complicated inequalities for the energies. However, when all three final particles are massless, the kinematic restrictions become simplify to

$$E_1,E_2,E_3\;\le\;\tfrac{1}{2}E_{\rm tot}\,,\qquad E_1\,+E_2\,+E_3\;=\;E_{\rm tot}\,. \tag{S.15}$$

Problem **2**:

We are now ready to calculate the muon decay rate and the electons' spectrum. According to the general rules of decay

$$d\Gamma \;=\; \frac{|\mathcal{M}|^2}{2M}\, d\mathcal{P}\,, \tag{S.16}$$

thus in light of eqs. (2) and (S.8),

$$d\Gamma(\mu^- \to e^-\nu_\mu\bar\nu_e) \;=\; \frac{G_F^2}{8\pi^5 M_\mu}\,(p_\mu\cdot p_{\bar\nu})(p_e\cdot p_\nu)\times dE_e\, dE_\nu\, dE_{\bar\nu}\, d^3\Omega\,\delta(E_e+E_\nu+E_{\bar\nu}-M_\mu). \tag{S.17}$$

In the muon's frame,

$$(p_\mu\cdot p_{\bar\nu}) \;=\; M_\mu E_{\bar\nu} \tag{S.18}$$

while

$$(p_e\cdot p_e) \;=\; E_eE_\nu \;-\; p_e p_\nu\cos\theta_{e\nu} \;=\; E_eE_\nu \;+\; \tfrac12 p_e^2 \;+\; \tfrac12 p_\nu^2 \;-\; \tfrac12 p_{\bar\nu}^2; \tag{S.19}$$

neglecting the electron and neutrino's masses, we may rewrite (S.19) as

$$\tfrac12(E_e+E_\nu)^2 \;-\; \tfrac12 E_{\bar\nu}^2 \;=\; \tfrac12 M_\mu(M_\mu-2E_{\bar\nu}). \tag{S.20}$$

Consequently,

$$d\Gamma(\mu^- \to e^-\nu_\mu\bar\nu_e) \;=\; \frac{G_F^2}{16\pi^5}\, M_\mu E_{\bar\nu}(M_\mu-2E_{\bar\nu})\times dE_e\, dE_\nu\, dE_{\bar\nu}\, d^3\Omega\,\delta(E_e+E_\nu+E_{\bar\nu}-M_\mu). \tag{S.21}$$

At this point we are ready to integrate over the final-state variables. In light of $\int d^3\Omega = 8\pi^2$ and the kinematic limits (S.15), we immediately obtain

$$\begin{aligned}
\Gamma(\mu^- \to e^-\nu_\mu\bar\nu_e) \;&=\; \frac{G_F^2 M_\mu}{2\pi^3}\iiint_0^{\frac12 M_\mu} dE_e\, dE_{\bar\nu}\, dE_\nu\, E_{\bar\nu}(M_\mu-2E_{\bar\nu})\delta(E_e+E_\nu+E_{\bar\nu}-M_\mu)\\
&=\; \frac{G_F^2 M_\mu}{2\pi^3}\int_0^{\frac12 M_\mu} dE_e \int_{\frac12 M_\mu - E_e}^{\frac12 M_\mu} dE_{\bar\nu}\, E_{\bar\nu}(M_\mu-2E_{\bar\nu})\\
&=\; \frac{G_F^2 M_\mu}{2\pi^3}\int_0^{\frac12 M_\mu} dE_e E_e^2(\tfrac12 M_\mu - \tfrac23 E_e).
\end{aligned} \tag{S.22}$$

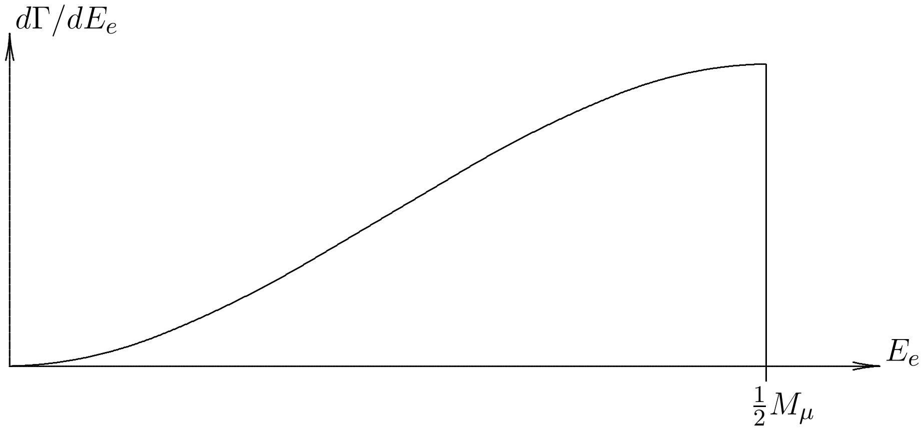
In other words, the partial muon decay rate with respect to the final electron's energy is given by

$$\frac{d\Gamma}{dE_e} = \frac{G_F^2 M_\mu}{12\pi^3} \times E_e^2(3M_\mu - 4E_e) \tag{S.23}$$

or rather

$$\frac{d\Gamma}{dE_e} \approx \begin{cases} \frac{G_F^2}{12\pi^3} M_\mu E_e^2(3M_\mu - 4E_e) & \text{for } E_e < \frac{1}{2}M_\mu, \\ 0 & \text{for } E_e > \frac{1}{2}M_\mu. \end{cases} \tag{S.24}$$

Graphically,



Note how this curve smoothly reaches its maximim at $E_e = \frac{1}{2}M_\mu$ and then abruptly falls down to zero.

It remains to calculate the total decay rate of the muon by integrating the partial rate (S.24) over the electron's energy. The result is

$$\Gamma_{\text{tot}}(\mu \to e\nu\bar{\nu}) = \frac{G_F^2 M_\mu^5}{192\pi^3}\,. \tag{S.25}$$