Problem **6.1**:

We are interested in a tree-level (in QED) elastic scattering of an electron off a proton via a virtual photon exchange. The amplitude for this process has form

$$\mathcal{M} \;=\; \frac{j_\mu(e^-)\,j^\mu(p^+)}{q^2} \tag{S.1}$$

where $q = p' - p = k - k'$ is the momentum of the virtual photon,

$$j_\mu(e^-) \;=\; -e\,\bar{u}(k')\gamma^\mu u(k) \tag{S.2}$$

is the electron's tree-level electromagnetic current while the proton's current (and hence the vertex) has non-trivial form factors due to strong interactions:

$$\begin{aligned} j^\mu(p^+) \;&=\; +e\,\bar{u}(p')\left[\gamma^\mu\,F_1(q^2) \;+\; \frac{i\sigma^{\mu\nu}q_\nu}{2M}\,F_2(q^2)\right]u(p) \\ &=\; +e\,\bar{u}(p')\left[\gamma^\mu\,(F_1+F_2)(q^2) \;-\; \frac{(p+p')^\mu}{2M}\,F_2(q^2)\right]u(p). \end{aligned} \tag{S.3}$$

We need to sum / average the mod-squared amplitude (S.1) over the electron's and proton's spin states, thus

$$\begin{aligned} \frac{1}{4}\sum_{\substack{\text{all}\\ \text{spins}}}|\mathcal{M}|^2 \;&=\; \frac{1}{4(q^2)^2}\times\sum_{\substack{\text{electron's}\\ \text{spins}}} j^\mu(e^-)j^\nu(e^-)\times\sum_{\substack{\text{proton's}\\ \text{spins}}} j^\mu(p^+)j^\nu(p^+) \\ &=\; \frac{e^2}{(q^2)^2}\left[k'_\mu k_\nu \;+\; k_\mu k'_\nu \;-\; g_{\mu\nu}(kk')\right]\times\sum_{\substack{\text{proton's}\\ \text{spins}}} j^\mu(p^+)j^\nu(p^+) \end{aligned} \tag{S.4}$$

where the second eqality — the sum over the electron's spin states — should be familiar by now; for simplicity, I neglected the electron's mass in thus sum.

Summing over the proton's spin states is more complicated:

$$\begin{aligned}
\sum_{\substack{\text{proton's}\\\text{spins}}} j^\mu(p^+)j^\nu(p^+) \;&=\; e^2\ \mathrm{tr}\left\{\left[\gamma^\mu\,(F_1+F_2)-\frac{(p+p')^\mu}{2M}\,F_2\right](\not{p}+M)\right.\\
&\qquad\qquad \left.\left[\gamma^\nu\,(F_1+F_2)-\frac{(p+p')^\nu}{2M}\,F_2\right](\not{p}'+M)\right\}\\
&=\; e^2(F_1+F_2)^2\ \mathrm{tr}\left[\gamma^\mu(\not{p}+M)\gamma^\nu(\not{p}'+M)\right]\\
&\quad+\; e^2F_2^2\,\frac{(p+p')^\mu(p+p')^\nu}{4M^2}\ \mathrm{tr}\left[(\not{p}+M)(\not{p}'+M)\right]\\
&\quad-\; e^2(F_1+F_2)F_2\left\{\frac{(p+p')^\mu}{2M}\ \mathrm{tr}\left[(\not{p}+M)\gamma^\nu(\not{p}'+M)\right]\;+\;(\mu\leftrightarrow\nu)\right\}\\
&=\; e^2(F_1+F_2)^2\times 4\left[p^\mu p'^\nu+p'^\mu p^\nu+g^{\mu\nu}(M^2-pp')\right]\\
&\quad+\; e^2F_2^2\times\frac{(p+p')^\mu(p+p')^\nu}{4M^2}\times 4[pp'+M^2]\\
&\quad-\; 2e^2(F_1+F_2)F_2\times 2[(p+p')^\mu(p+p')^\nu]
\end{aligned}$$

⟨⟨using $p^\mu p'^\nu+p'^\mu p^\nu \;=\; \frac{1}{2}(p+p')^\mu(p+p')^\nu \;-\; \frac{1}{2}q^\mu q^\nu$ ⟩⟩

$$\begin{aligned}
&=\; e^2(F_1+F_2)^2\times\left[-2q^\mu q^\nu\;+\;4g^{\mu\nu}(M^2-pp')\right]\\
&\quad+\; e^2(p+p')^\mu(p+p')^\nu\times\left[2(F_1+F_2)^2\;+\;F_2^2\left(1+\frac{pp'}{M^2}\right)\;-\;4(F_1+F_2)F_2\right]
\end{aligned}$$

⟨⟨using $q^2 \;=\; 2M^2\;-\;2pp'$ ⟩⟩

$$\begin{aligned}
&=\; 2e^2(F_1+F_2)^2\times\left(q^2g^{\mu\nu}\;-\;q^\mu q^\nu\right)\\
&\quad+\; 2e^2(p+p')^\mu(p+p')^\nu\times\left(F_1^2\;-\;\frac{q^2}{4M^2}\,F_2^2\right).
\end{aligned}\tag{S.5}$$

Substituting eq. (S.5) back into eq. (S.4) now gives us

$$\frac{1}{4}\sum_{\substack{\text{all}\\\text{spins}}}|\mathcal{M}|^2 \;=\; \frac{2e^4}{(q^2)^2}\left[\mathcal{A}\left(F_1^2-\tfrac{q^2}{4M^2}F_2^2\right)\;+\;\mathcal{B}\,(F_1+F_2)^2\right]\tag{S.6}$$

where

$$\begin{aligned}\mathcal{A} &= [k'_\mu k_\nu + k_\mu k'_\nu - g_{\mu\nu}(kk')] \times (p+p')^\mu (p+p')^\nu \\ &= 2(pk+p'k)(pk'+p'k') \;-\; (kk')(p+p')^2 \\ &\text{or in terms of Mandelstam's kinematic variables } s,\ t \text{ and } u, \\ &= \tfrac{1}{2}(s-u)^2 \;+ \tfrac{1}{2}t(4M^2-t) \\ &= 2(s-2M^2)^2 \;+\; 2st\end{aligned} \tag{S.7}$$

and

$$\begin{aligned}\mathcal{B} &= [k'_\mu k_\nu + k_\mu k'_\nu - g_{\mu\nu}(kk')] \times \left[q^2 g^{\mu\nu} - q^\mu q^\nu\right] \\ &= (2-4+1)\times(kk')q^2 \;-\; 2(kq)(k'q) \\ &= -(\tfrac{1}{2}t)(t) \;-\; 2(\tfrac{1}{2}t)(-\tfrac{1}{2}t) \;=\; -t^2.\end{aligned} \tag{S.8}$$

In the lab frame,

$$s \;=\; M^2 \;+\; 2ME, \qquad t \;=\; q^2 \;=\; -2EE'(1-\cos\theta) \tag{S.9}$$

and (*cf.* discussion of the Compton effect in the textbook)

$$E' \;=\; \frac{ME}{M+E(1-\cos\theta)}\,. \tag{S.10}$$

Also,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm lab} \;=\; \left(\frac{E'}{8\pi ME}\right)^2 \times \tfrac{1}{4}\sum_{\substack{\rm all\\ \rm spins}} |\mathcal{M}|^2\,. \tag{S.11}$$

Combining this formulæ with eq. (S.6) gives us

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm lab} \;=\; \frac{\alpha^2}{8M^2E^4(1-\cos\theta)^2} \times \left[\mathcal{A}\left(F_1^2 - \tfrac{q^2}{4M^2}F_2^2\right) \;+\; \mathcal{B}\,(F_1+F_2)^2\right] \tag{S.12}$$

where we now evaluate eqs. (S.7) and (S.8) as

$$\begin{aligned}\mathcal{A} &= \frac{4M^3E^2(1+\cos\theta)}{M+E(1-\cos\theta)}\,, \\ \mathcal{B} &= -(q^2)^2 \;=\; q^2 \times \frac{2ME^2(1-\cos\theta)}{M+E(1-\cos\theta)}\,.\end{aligned} \tag{S.13}$$

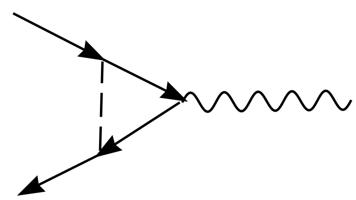
Substituting eqs. (S.13) into eq. (S.12) finally gives us the *Rosenbluth formula*

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = \frac{\alpha^2}{2E^2(1-\cos\theta)^2} \times \frac{\left[(1+\cos\theta)\,\left(F_1^2 - \frac{q^2}{4M^2}F_2^2\right) \;+\; (1-\cos\theta)\,\frac{q^2}{2M^2}(F_1+F_2)^2\right]}{1 \,+\, \frac{E}{M}(1-\cos\theta)}\,. \tag{S.14}$$

$\mathcal{Q.E.D.}$

Problem **6.3**(a):

The leading-order effect of the Higgs boson on the electron's magnetic moment comes from the following Feynman diagram:



In terms of the $e\gamma$ vertex correction, we have

$$ie\,\bar{u}\big(\delta\Gamma^{\mu}_{\text{amp}}\big)u \;=\; \int\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m_h^2+i\epsilon}\;\times\;\bar{u}(p')\,\frac{-i\lambda}{\sqrt{2}}\,\frac{i}{\not{p}'+\not{k}-m_e+i\epsilon}\,(ie\gamma^{\mu})\,\frac{i}{\not{p}+\not{k}-m_e+i\epsilon}\,\frac{-i\lambda}{\sqrt{2}}\,u(p), \tag{S.15}$$

which can be evaluated in exactly the same way as the similar QED correction with a virtual photon. Using Feynman parameters $x+y+z=1$ for the three denominators, we have

$$\delta\Gamma^{\mu}_{\text{amp}} \;=\; \iiint dxdydz\,\delta(x+y+z-1)\int\frac{d^4\ell}{(2\pi)^4}\,\frac{i\lambda^2\,\mathcal{N}^{\mu}}{(\ell^2-\Delta+i\epsilon)^3} \tag{S.16}$$

where

$$\ell \;=\; k \;+\; xp \;+\; yp', \qquad \Delta \;=\; zm_h^2 \;+\; (1-z)^2m_e^2 \;-\; xyq^2 \tag{S.17}$$

and

$$\begin{aligned}\mathcal{N}^{\mu} \;&=\; (\not{p}'+\not{k}+m_e)\,\gamma^{\mu}\,(\not{p}+\not{k}+m_e)\\ &\cong\; \left[(1+z)^2m_e^2+xyq^2-\tfrac{1}{2}\ell^2\right]\times\gamma^{\mu} \;+\; 2m_e^2(1-z^2)\times\frac{i\sigma^{\mu\nu}q_{\nu}}{2m_e}\,.\end{aligned} \tag{S.18}$$

The second expression here follows from the first after a few tricks you should have learned from the QED calculation. As in QED, the first term on the second line of eq. (S.18) contributes to the $\delta F_1(q^2)$ while the second term contributes to the $\delta F_2(q^2)$, and hence to the anomalous magnetic moment. Thus,

$$\begin{aligned}\delta_{\rm Higgs}F_2(q^2) &= \iiint dxdydz\,\delta(x+y+z-1)\left\{2m_e^2\lambda^2(1-z^2)\int\frac{d^4\ell}{(2\pi)^4}\,\frac{i}{(\ell^2-\Delta+i\epsilon)^3}\right\}\\ &= \frac{\lambda^2}{16\pi^2}\iiint dxdydz\,\delta(x+y+z-1)\,\frac{m_e^2(1-z^2)}{\Delta = zm_h^2+(1-z)^2m_e^2-xyq^2}\end{aligned}\qquad\text{(S.19)}$$

and therefore

$$\begin{aligned}\delta_{\rm Higgs}\left(\frac{g-2}{2}\right) &= \delta_{\rm Higgs}F_2(q^2=0)\\ &= \frac{\lambda^2}{16\pi^2}\iiint dxdydz\,\delta(x+y+z-1)\,\frac{m_e^2(1-z^2)}{zm_h^2+(1-z)^2m_e^2}\\ &= \frac{\lambda^2}{16\pi^2}\int_0^1 dz\,\frac{(1-z)(1-z^2)m_e^2}{zm_h^2+(1-z)^2m_e^2}\\ &= \frac{\lambda^2m_e^2}{16\pi^2m_h^2}\left[\log\frac{m_h^2}{m_e^2}\;-\;\frac{7}{6}\;+\;O\left(\frac{m_e^2}{m_h^2}\right)\right].\end{aligned}\qquad\text{(S.20)}$$

Problem **6.3**(b):

According to eq. (S.20), the effect of the virtual Higgs boson one the electron's magnetic moment is suppressed by two very small factors, namely $(m_e/m_h)^2$ and $\lambda^2 = (m_e/\langle H\rangle)^2$ where $\langle H\rangle \approx 245$ GeV is the Higgs field's vacuum expectation value that gives rise to the electron's mass in the first place. Presently, the Higgs mass m_h is not known, but there is experimental lower limit $m_h \gtrsim 100$ GeV. Hence,

$$\begin{aligned}\delta_{\rm Higgs}\left(\frac{g-2}{2}\right)_e &\lesssim 1.7\cdot 10^{-23},\\ \delta_{\rm Higgs}\left(\frac{g-2}{2}\right)_\mu &\lesssim 1.7\cdot 10^{-14},\end{aligned}\qquad\text{(S.21)}$$

both correction being much smaller than the present-day experimental uncertainties of the two leptons' anomalous magnetic moments.

Problem **6.3**(c):

The axion's contribution to the anomalous magnetic moment can be calculated similarly to that of the Higgs boson. Again, the $e\gamma$ vertex correction is given by eq. (S.16), only now

$$\begin{aligned} \mathcal{N}^\mu &= -\gamma^5(\not{p}'+\not{k}+m_e)\,\gamma^\mu\,(\not{p}+\not{k}+m_e)\gamma^5 \\ &\cong \left[(1-z)^2 m_e^2 + xyq^2 - \tfrac{1}{2}\ell^2\right] \times \gamma^\mu \;-\; 2m_e^2(1-z)^2 \times \frac{i\sigma^{\mu\nu}q_\nu}{2m_e}\,. \end{aligned} \tag{S.22}$$

Consequently,

$$\delta_{\text{axion}}\left(\frac{g-2}{2}\right) \;=\; -\frac{\lambda^2}{16\pi^2}\int_0^1 dz\,\frac{(1-z)^3\,m_e^2}{zm_a^2+(1-z)^2m_e^2} \;=\; -\frac{\lambda^2}{16\pi^2}\,I(m_a^2/m_e^2),$$

where the analytic formula for the integral $I(m_a^2/m_e^2)$ is too complicated to display here. Asymptotically, $I\approx\frac{1}{2}$ for a light axion ($m_a\ll m_e$) and $I\approx\frac{m_e^2}{m_a^2}\left[\log(\frac{m_a^2}{m_e^2})-\frac{11}{6}\right]\ll 1$ for a heavy axion ($m_a\gg m_e$).

Given the current experimental limit $|\delta\left(\frac{g-2}{2}\right)_e| < 10^{-11}$, there is a rather low experimental limit on the electron-axion coupling: A light axion must have $|\lambda| < 4\cdot 10^{-5}$; for a heavier axion, $|\lambda| < 4\cdot 10^{-5}/\sqrt{2I}$.