**Problem 1**:

First of all, we need the Feynman rules. The Scalar QED has a charged scalar field $\Phi$, the EM field $A^\mu$ and the bare Lagrangian

$$\mathcal{L}_{\text{bare}} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\Phi^* D^\mu\Phi - m_0^2\Phi^*\Phi - \tfrac{1}{4}\lambda_0(\Phi^*\Phi)^2 \tag{S.1}$$

where for the scalar field of charge $-e$,

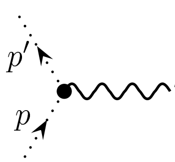
$$D_\mu\Phi^* D^\mu\Phi \equiv \partial^\mu\Phi^*\,\partial^\mu\Phi + e_0A^\mu\left(-i\Phi^*\,\partial_\mu\Phi + i\partial_\mu\Phi^*\,\Phi\right) + e_0^2A^\mu A_\mu\Phi^*\Phi. \tag{S.2}$$

The Feynman rules follow from the above Lagrangian. In the bare-field picture, we have the (bare) propagators

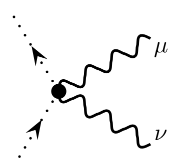
$\Phi$  $\Phi^* \quad = \quad \dfrac{i}{p^2 - m_0^2 + i0}$

$A^\mu$  $A^\nu \quad = \quad \dfrac{i}{k^2+i0}\left(-g^{\mu\nu} + (1-\xi)\dfrac{k^\mu k^\nu}{k^2}\right)$ (S.3)

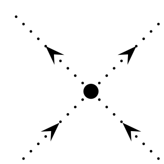
and the (bare) vertices



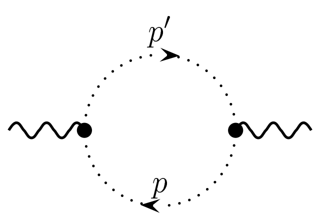
$= ie_0(p+p')^\mu,$

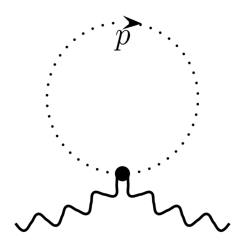


$= -2ie_0^2g^{\mu\nu},$


$= -i\lambda_0\,.$ (S.4)

At this point, we can see that there are indeed two one-loop Feynman diagrams contributing to the 1PI two-photon amplitude, namely



and


, (S.5)

which respectively yield

$$
\begin{aligned}
-i\Sigma^{\text{first}}_{\alpha\beta} &= \int \frac{d^4p}{(2\pi)^4}\,[ie_0(p+p')_\alpha]\,\frac{i}{p^2-m_0^2+i0}\,[ie_0(p+p')_\alpha]\,\frac{i}{p'^2-m_0^2+i0}\,,\\
-i\Sigma^{\text{second}}_{\alpha\beta} &= \int \frac{d^4p}{(2\pi)^4}\,[-2e_0^2 i g_{\alpha\beta}]\,\frac{i}{p^2-m_0^2+i0}\,.
\end{aligned}
\tag{S.6}
$$

For the first diagram, we have $p' = p + k$ and hence

$$
\begin{aligned}
\Sigma^{\text{first}}_{\alpha\beta}(k) &= \int \frac{d^4p}{(2\pi)^4}\,\frac{ie_0^2\,(2p+k)_\alpha(2p+k)\beta}{\big(p^2-m_0^2+i0\big)\big((p+k)^2-m_0^2+i0\big)}\\
&= ie_0^2\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\big(2\ell+(1-2x)k\big)_\alpha\big(2\ell+(1-2x)k\big)_\beta}{\big(\ell^2-\Delta+i0\big)^2}
\end{aligned}
\tag{S.7}
$$

where as usual $\ell = p + xk$ and $\Delta = m_0^2 - x(1-x)k^2$. The numerator in the second integral is equivalent to

$$
\begin{aligned}
\big(2\ell+(1-2x)k\big)_\alpha\big(2\ell+(1-2x)k\big)_\beta &\cong 4\ell_\alpha\ell_\beta \;+\; (1-2x)^2k_\alpha k_\beta\\
&\cong \frac{4\ell^2}{D}\,g_{\alpha\beta} \;+\; (1-2x)^2k_\alpha k_\beta\\
&= \big(\tfrac{4}{D}\ell^2 \;+\; (1-2x)^2k^2\big)\,g_{\alpha\beta}\\
&\quad+\;(1-2x)^2\,\big(k_\alpha k_\beta \;-\; g_{\alpha\beta}k^2\big)
\end{aligned}
\tag{S.8}
$$

and hence

$$
\Sigma^{\text{first}}_{\alpha\beta}(k) \;=\; \big(k_\alpha k_\beta \;-\; g_{\alpha\beta}k^2\big)\times\mathcal{B}^{(1)}(k^2) \;+\; g_{\alpha\beta}\times\mathcal{F}^{(1)}(k^2)
\tag{S.9}
$$

where

$$
\mathcal{B}^{(1)}(k^2) \;=\; e_0^2\int_0^1 dx\,(1-2x)^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{i}{\big(\ell^2-\Delta+i0\big)^2}
\tag{S.10}
$$

and

$$
\mathcal{F}^{(1)}(k^2) \;=\; e_0^2\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\,\frac{i\big(\frac{4}{D}\ell^2 \;+\; (1-2x)^2k^2\big)}{\big(\ell^2-\Delta+i0\big)^2}\,.
\tag{S.11}
$$

Our first task is to verify the tensor structure of the two-photon amplitude, so let us focus on the coefficient $\mathcal{F}$ of the wrong tensor. Applying Wick rotation and dimensional regularization to the momentum integral in eq. (S.11), we calculate

$$\begin{aligned}
\int\frac{d^4\ell}{(2\pi)^4}\,\frac{i\bigl(\frac{4}{D}\ell^2\,+\,(1-2x)^2k^2\bigr)}{\bigl(\ell^2-\Delta+i0\bigr)^2}\ &=\ \mu^{4-D}\int\frac{d^D\ell_E}{(2\pi)^D}\,\frac{(1-2x)^2k^2\,-\,\frac{4}{D}\ell_E^2}{(\ell_E^2+\Delta)^2}\\
&=\ \mu^{4-D}\int\limits_0^\infty\!dt\,t\int\frac{d^D\ell_E}{(2\pi)^D}\,\bigl((1-2x)^2k^2\,-\,\tfrac{4}{D}\ell_E^2\bigr)e^{-t(\ell_E^2+\Delta)}\\
&=\ \mu^{4-D}\int\limits_0^\infty\!dt\,te^{-t\Delta}\left((1-2x)^2k^2\,+\,\frac{4}{D}\frac{\partial}{\partial t}\right)\int\frac{d^D\ell_E}{(2\pi)^D}\,e^{-t\ell_E^2}\\
&=\ \frac{\mu^{4-D}}{(4\pi)^{D/2}}\int\limits_0^\infty\!dt\,te^{-t\Delta}\left((1-2x)^2k^2\times t^{-D/2}\,-\,2t^{-(D/2)-1}\right)\\
&=\ \frac{\mu^{4-D}}{(4\pi)^{D/2}}\Bigl((1-2x)^2k^2\times\Gamma\left(2-\tfrac{D}{2}\right)\Delta^{\frac{D}{2}-2}\\
&\qquad\qquad\qquad -2\Gamma\left(1-\tfrac{D}{2}\right)\Delta^{\frac{D}{2}-1}\Bigr)\\
&=\ \frac{\mu^{4-D}}{(4\pi)^{D/2}}\,\Gamma\left(1-\tfrac{D}{2}\right)\times\frac{\partial}{\partial x}\left((1-2x)\Delta^{\frac{D}{2}-1}\right)
\end{aligned}\tag{S.12}$$

where the last equality follows from

$$\frac{\partial\Delta}{\partial x}\ =\ -(1-2x)k^2\qquad\text{and}\qquad\left(\tfrac{D}{2}-1\right)\Gamma\left(1-\tfrac{D}{2}\right)\ =\ -\Gamma\left(2-\tfrac{D}{2}\right).$$

Consequently, the integrand of the $\int dx$ integral in eq. (S.11) is a total derivative and hence

$$\begin{aligned}
\mathcal{F}^{(1)}(k^2)\ &=\ \frac{e_0^2\mu^{4-D}}{(4\pi)^{D/2}}\,\Gamma\left(1-\tfrac{D}{2}\right)\times\left((1-2x)\Delta^{\frac{D}{2}-1}\right)_{x=0}^{x=1}\\
&=\ \frac{e_0^2\mu^{4-D}}{(4\pi)^{D/2}}\,\Gamma\left(1-\tfrac{D}{2}\right)\times\left(-(m_0^2)^{\frac{D}{2}-1}\,-\,(m_0^2)^{\frac{D}{2}-1}\right)\\
&=\ -2\Gamma\left(1-\tfrac{D}{2}\right)\,e_0^2m_0^{D-2}\mu^{4-D}(4\pi)^{-D/2}.
\end{aligned}\tag{S.13}$$

Note that thanks to $\Delta(x\,=\,1)\,=\,\Delta(x\,=\,0)\,=\,m_0^2$, the right hand side of eq. (S.13) is independent of the photon's momentum $k$. Since the second diagram's contribution $\Sigma_{\alpha\beta}^{\text{second}}$

is also $k$–independent, this allows for the cancellation of the wrong tensor structure of the two-photon amplitude between the two diagrams.

Indeed, for the second diagram we have

$$\Sigma^{\text{second}}_{\alpha\beta} \;=\; g_{\alpha\beta} \times \mathcal{F}^{(2)} \tag{S.14}$$

where

$$\begin{aligned}
\mathcal{F}^{(2)} &= \int\frac{d^4p}{(2\pi)^4}\,\frac{2ie_0^2}{p^2-m^2+i0} \;=\; \int\frac{d^4p_E}{(2\pi)^4}\,\frac{2e_0^2}{p_E^2+m_0^2}\\
&\longrightarrow\; 2e_0^2\int\frac{d^Dp_E}{(2\pi)^D}\,\frac{\mu^{4-D}}{p_E^2+m_0^2}\\
&= 2e_0^2\mu^{4-D}\int\limits_0^\infty dt\,e^{-tm_0^2}\times(4\pi t)^{-D/2}\\
&= 2e_0^2\mu^{4-D}(4\pi)^{-D/2}\times\Gamma\left(1-\tfrac{D}{2}\right)m_0^{D-2}\\
&= -\mathcal{F}^{(1)},
\end{aligned} \tag{S.15}$$

which means that

$$\Sigma^{1\,\text{loop}}_{\alpha\beta}(k) \;=\; \Sigma^{\text{first}}_{\alpha\beta}(k) \;+\; \Sigma^{\text{second}}_{\alpha\beta} \;=\; \left(k_\alpha k_\beta \;-\; g_{\alpha\beta}k^2\right)\times\mathcal{B}^{(1)}(k^2) \;+\; g_{\alpha\beta}\times 0. \tag{S.16}$$

This completes part (a) of the problem.

Our next task is to calculate the $\mathcal{B}^{1\,\text{loop}} = \mathcal{B}^{(1)}$ as a function of $k^2$. The momentum integral in eq. (S.10) should be rather familiar after so much related class- and home-work, so let me simply write down the result:

$$\begin{aligned}
\int\frac{d^4\ell}{(2\pi)^4}\,\frac{i}{\left(\ell^2-\Delta+i0\right)^2} \;&\underset{DR}{\longrightarrow}\; \int\frac{d^D\ell_E}{(2\pi)^D}\,\frac{-\mu^{4-D}}{(\ell_E^2+\Delta)^2}\\
&= \frac{-1}{16\pi^2}\,\Gamma\left(2-\tfrac{D}{2}\right)\left(\frac{4\pi\mu^2}{\Delta}\right)^{2-(D/2)}\\
&\underset{\epsilon\to0}{\longrightarrow}\; \frac{-1}{16\pi^2}\left(\frac{1}{\bar\epsilon}\;+\;\log\frac{\mu^2}{\Delta}\right),
\end{aligned} \tag{S.17}$$

and consequently,

$$\mathcal{B}^{1\,\text{loop}}(k^2) \;=\; -\frac{e_0^2}{16\pi^2}\int_0^1 dx\,(1-2x)^2\,\left(\frac{1}{\bar\epsilon}\;+\;\log\frac{\mu^2}{\Delta}\right). \tag{S.18}$$

It remains to substitute $\Delta(x)=m^2-x(1-x)k^2$ where the difference between the physical scalar mass $m$ and the bare mass $m_0$ can be ignored as being $O(\alpha)$. Thus,

$$\mathcal{B}^{1\,\text{loop}}(k^2=0) \;=\; -\frac{e_0^2}{16\pi^2}\int_0^1 dx\,(1-2x)^2\,\left(\frac{1}{\bar\epsilon}\;+\;\log\frac{\mu^2}{m^2}\right) \;=\; -\frac{e_0^2}{16\pi^2}\times\frac{1}{6}\left(\frac{1}{\bar\epsilon}\;+\;\log\frac{\mu^2}{m^2}\right) \tag{S.19}$$

and

$$\begin{aligned}
\mathcal{B}^{1\,\text{loop}}(k^2)\;-\;\mathcal{B}^{1\,\text{loop}}(0) \;&=\; -\frac{e_0^2}{16\pi^2}\int_0^1 dx\,(1-2x)^2\,\log\frac{m^2}{m^2-x(1-x)k^2}\\
&\xrightarrow[-k^2\gg m^2]{}\; +\frac{e_0^2}{16\pi^2}\int_0^1 dx\,(1-2x)^2\left(\log\frac{-k^2}{m^2}\;+\;\log x(1-x)\right)\\
&=\;\frac{e_0^2}{16\pi^2}\left(\frac{1}{6}\log\frac{-k^2}{m^2}\;-\;\frac{4}{9}\right)\\
&=\;\frac{\alpha_0}{12\pi}\left(\log\frac{-k^2}{m^2}\;-\;\frac{8}{3}\right).
\end{aligned} \tag{S.20}$$

It remains to draw conclusions from eqs. (S.19) and (S.20). As explained in class,

$$\begin{aligned}
Z_3 \;&=\; \frac{1}{1-\mathcal{B}(0)} \;=\; 1\;+\;\mathcal{B}^{1\,\text{loop}}(0)\;+\;O(\alpha^2)\\
&=\; 1\;-\;\frac{\alpha_0}{12\pi}\left(\frac{1}{\bar\epsilon}\;+\;\log\frac{\mu^2}{m^2}\right)\;+\;O(\alpha^2),
\end{aligned} \tag{S.21}$$

while $e_{\text{phys}}=e_0\sqrt{Z_3}$ and hence

$$\frac{1}{\alpha_{\text{phys}}} \;=\; \frac{1}{\alpha_0}\;-\;\frac{1}{\alpha_0}\mathcal{B}(0) \;=\; \frac{1}{\alpha_0}\;+\;\frac{1}{12\pi}\left(\frac{1}{\bar\epsilon}\;+\;\log\frac{\mu^2}{m^2}\right)\;+\;O(\alpha_0). \tag{S.22}$$

Finally, the effective QED coupling at high energies is given by

$$\begin{aligned}\frac{1}{\alpha_{\text{eff}}(k^2)} &= \frac{1}{\alpha_0} - \frac{1}{\alpha_0}\mathcal{B}(k^2) = \frac{1}{\alpha_{\text{eff}}(0)} - \frac{1}{\alpha_0}\left[\mathcal{B}(k^2) - \mathcal{B}(0)\right] \\ &\approx \frac{1}{\alpha_{\text{eff}}(0)} - \frac{1}{12\pi}\left(\log\frac{-k^2}{m^2} - \frac{8}{3}\right)\end{aligned} \tag{S.23}$$

where $\alpha_{\text{eff}}(0)$ is the same as $\alpha_{\text{phys}}$ in eq. (S.22).