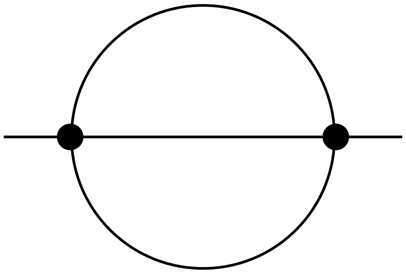

Problem **1**:

At the two-loop level, there is just one p-dependent 1PI Feynman diagram with two external legs, namely



hence

$$i\Sigma_{2\,\mathrm{loop}}(p^2) \;=\; \frac{(-i\lambda)^2}{6}\iint\frac{d^4k_1\,d^4k_2}{(2\pi)^8}\,\frac{i}{k_1^2-m^2+i0}\,\frac{i}{k_2^2-m^2+i0}\,\frac{i}{(k_1+k_2+p)^2-m^2+i0} \;+\; \mathrm{const.} \tag{S.1}$$

With the help of Feynman parameters x, y and z, we simplify

$$\prod\text{propagators} \;=\; \iiint dxdydz\,\delta(x+y+z-1)\,\frac{2i^3}{\mathcal{D}^3} \tag{S.2}$$

where

$$\begin{aligned}\mathcal{D} \;&=\; xk_1^2 \;+\; yk_2^2 \;+\; z(k_1+k_2+p)^2 \;-\; m^2 \;+\; i0\\ &=\; \alpha\ell_1^2 \;+\; \beta\ell_2^2 \;+\; \gamma p^2 \;-\; m^2 \;+\; i0\end{aligned} \tag{S.3}$$

for

$$\begin{aligned}\ell_1 \;&=\; k_1 \;+\; \frac{z}{x+z}\,(k_2+p),\\ \ell_2 \;&=\; k_2 \;+\; \frac{xz}{xy+xz+yz}\,p,\\ \alpha \;&=\; x+z,\\ \beta \;&=\; \frac{xy+xz+yz}{x+z}\,,\\ \gamma \;&=\; \frac{xyz}{xy+xz+yz}\,.\end{aligned} \tag{S.4}$$

Note that the Jacobian $d(\ell_1,\ell_2)/(dk_1,dk_1)=1$, thus

$$\Sigma_{2\,\mathrm{loop}}(p^2) \;=\; \frac{\lambda^2}{6}\iiint dxdydz\,\delta(x+y+z-1)\iint\frac{d^4\ell_1\,d^4\ell_2}{(2\pi)^8}\,\frac{2}{\left[\alpha\ell_1^2+\beta\ell_2^2+\gamma p^2-m^2+i0\right]^3}$$
$$+\;\mathrm{const} \tag{S.5}$$

and hence

$$\frac{d\Sigma_{2\,\mathrm{loop}}}{dp^2} \;=\; \frac{\lambda^2}{6}\iiint dxdydz\,\delta(x+y+z-1)\,\gamma(x,y,z)\iint\frac{d^4\ell_1\,d^4\ell_2}{(2\pi)^8}\,\frac{-6}{\left[\alpha\ell_1^2+\beta\ell_2^2+\gamma p^2-m^2+i0\right]^4}\,. \tag{S.6}$$

Our next step is to Wick-rotate the loop momenta ℓ_1 and ℓ_2 into Euclidean space and dimensionally regularize. This gives us

$$\begin{aligned}
&\iint\frac{d^d\ell_1\,d^d\ell_2}{(2\pi)^{(2d)}}\,\frac{+6\,\mu^{4\epsilon}}{\left[\alpha(\ell_1^E)^2+\beta(\ell_2^E)^2-\gamma p^2+m^2\right]^4}\\
&\qquad = \int\limits_0^\infty d\rho\,\rho^3\,\mu^{4\epsilon}e^{-\rho(m^2-\gamma p^2)}\iint\frac{d^d\ell_1\,d^d\ell_2}{(2\pi)^(2d)}\,e^{-\rho\alpha(\ell_1^E)^2}e^{-\rho\beta(\ell_2^E)^2}\\
&\qquad = \int\limits_0^\infty d\rho\,\rho^3\,\mu^{4\epsilon}e^{-\rho(m^2-\gamma p^2)}\times(4\pi\rho\alpha)^{-d/2}\,\times(4\pi\rho\beta)^{-d/2}\\
&\qquad = \frac{\Gamma(2\epsilon)}{(4\pi)^4}\,(\alpha\beta)^{\epsilon-2}\left(\frac{4\pi\mu^2}{m^2-\gamma p^2}\right)^{2\epsilon}\\
&\qquad \xrightarrow[d\to4]{}\;\frac{1}{(4\pi)^4}\,\frac{1}{(\alpha\beta)^2}\left(\frac{1}{2\epsilon}\;-\;\gamma_E\;+\;\log\frac{4\pi\mu^2\sqrt{\alpha\beta}}{m^2-\gamma p^2}\right).
\end{aligned} \tag{S.7}$$

(Note the γ_E here is the Euler's constant and has nothing to do with the $\gamma(x,y,z)$, *cf.* eq. (S.4)).

Substituting this regularized momentum integral back into eq. (S.6), we arrive at

$$\begin{aligned}
\delta_Z^{(2)} &= -\left.\frac{d\Sigma_{2\,\text{loop}}}{dp^2}\right|_{p^2=m^2} \\
&= -\frac{\lambda^2}{1536\pi^4}\iiint dxdydz\,\delta(x+y+z-1)\,\frac{\gamma}{(\alpha\beta)^2} \\
&\qquad\times\left(\frac{1}{2\epsilon}\;-\;\gamma_E\;+\;\log\frac{4\pi\mu^2}{m^2}\;+\;\log\frac{\sqrt{\alpha\beta}}{1-\gamma}\right) \qquad\qquad (\text{S.8}) \\
&= -\frac{\lambda^2}{1536\pi^4}\iiint dxdydz\,\delta(x+y+z-1)\,\frac{xyz}{(xy+xz+yz)^3} \\
&\qquad\times\left(\frac{1}{2\epsilon}\;-\;\gamma_E\;+\;\log\frac{4\pi\mu^2}{m^2}\;+\;\tfrac{1}{2}\log\frac{(xy+xz+yz)^3}{(xy+xz+yz-xyz)^2}\right).
\end{aligned}$$

The remaining integral over the Feynman parameters is straightforward but tedious. To save your time, I have used Mathematica and gave you the results in eq. (4). Hence,

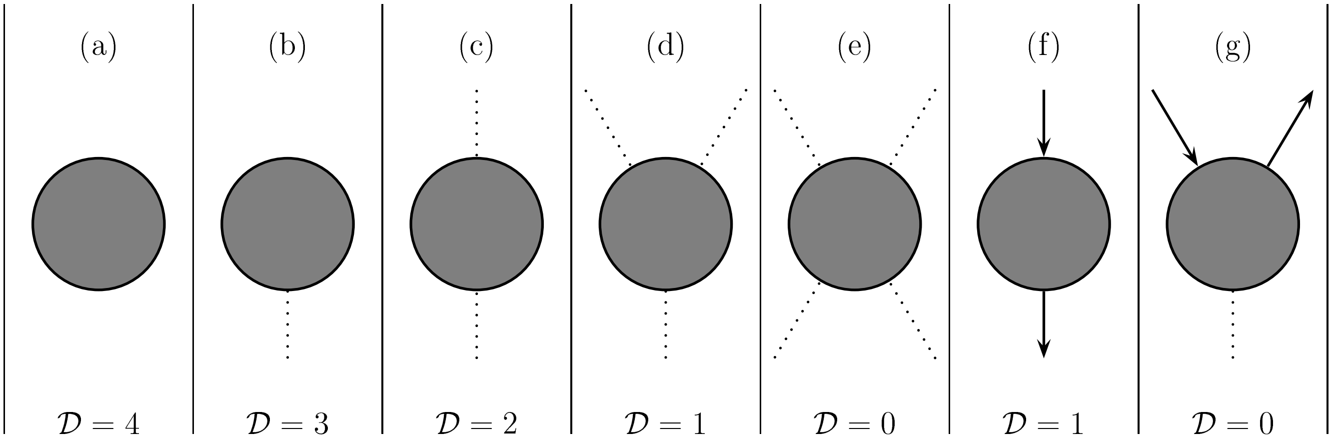
$$\delta_Z^{(2)} \;=\; -\frac{\lambda^2}{3072\pi^4}\left(\frac{1}{2\epsilon}\;-\;\gamma_E\;+\;\log\frac{4\pi\mu^2}{m^2}\;-\;\frac{3}{4}\right). \qquad (\text{S.9})$$

Problem **2**(a):

Concerning the superficial degree of divergence, the Yukawa theory works exactly like QED and has

$$\mathcal{D} \;=\; 4\;-\;E_\phi\;-\;\tfrac{3}{2}\,E_\psi\,. \qquad (\text{S.10})$$

Consequently, there are only seven superficially divergent amplitudes, namely



Furthermore, the amplitude (a) here is the vacuum energy while the amplitudes (b) and (d) vanish because of the parity symmetry. Indeed, the *pseudo*-scalar field Φ is parity-odd, hence the amplitudes involving odd number of pseudoscalar particles and no fermions must have parity-odd dependence on the particles' momenta. But to construct a parity-odd Lorentz-invariant combination of the Lorentz vectors $p_1^\alpha, p_2^\beta, \ldots$, one needs ϵ tensors, *e.g.* $\epsilon_{\alpha\beta\gamma\delta} p_1^\alpha p_2^\beta p_3^\gamma p_4^\delta$, which requires at least 4 linearly independent momenta (in $d = 4$ spacetime) and hence $n \geq 5$ external legs. For the amplitudes (b) and (d) involving one or three pseudoscalars only and no fermions, such construction is not available and the amplitudes vanish identically.

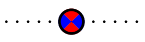
Unlike QED, the Yukawa theory does not give rise to Ward identities, so any 1PI amplitude that can diverge generally does. Hence, expanding the 1PI amplitudes (c), (e), (f), and (g) in powers of relevant momenta we find the following independent divergences:

(c) $\Sigma_\phi(p^2) = O(\Lambda^2) \times \text{const} + O(\log\Lambda) \times p^2 + \text{finite}$;

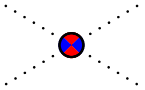
(e) $\mathcal{M}(s,t,u) = O(\log\Lambda) \times \text{const} + \text{finite}$;

(f) $\Sigma_\psi(\not{p}) = O(\Lambda^1) \times \text{const} + O(\log\Lambda) \times \not{p} + \text{finite}$;

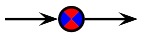
(g) $\Gamma^5(p', p) = \gamma^5 \times O(\log\Lambda) \times \text{const} + \text{finite}$. To cancel all these divergences *in situ* in the renormalized perturbation theory we need four counterterm-related Feynman vertices, namely



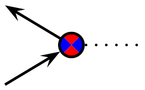
$$= -i\delta_m^\phi + ip^2\,\delta_Z^\phi\,,$$



$$= -i\delta_\lambda\,,$$



$$= -i\delta_m^\psi + i\not{p}\,\delta_Z^\psi\,, \tag{S.11}$$



$$= -\delta_g \gamma^5$$

Clearly, all these vertices follow from local (in x) counterterms in the renormalized Lagrangian, specifically

$$\mathcal{L}^{\text{counter}}_{\text{terms}} = \tfrac{1}{2}\delta_Z^\phi\,(\partial\Phi)^2 - \tfrac{1}{2}\delta_m^\phi\,\Phi^2 - \tfrac{1}{4!}\delta_\lambda\,\Phi^4 + i\delta_Z^\psi\,\overline{\Psi}\,\not{\partial}\Psi - \delta_m^\psi\,\overline{\Psi}\Psi - i\delta_g\,\Phi\overline{\Psi}\gamma^5\Psi. \tag{S.12}$$

In order to produce such counterterms, one starts from the bare Lagrangian

$$\mathcal{L}_{\text{bare}} = \tfrac{1}{2}(\partial\Phi_0)^2 - \tfrac{1}{2}m_0^2\Phi_0^2 - \tfrac{1}{4!}\lambda_0\Phi_0^4 + \overline{\Psi}_0(i\not{\partial} - M_0)\Psi_0 - ig_0\,\Phi_0\overline{\Psi}_0\gamma^5\Psi_0\,, \tag{S.13}$$

renormalizes the bare fields $\Phi_0(x)=\sqrt{Z_\phi}\Phi_r(x)$, $\Psi_0(x)=\sqrt{Z_\psi}\Psi_r(x)$, and splits into

$$\mathcal{L}_{\text{bare}} \;=\; \mathcal{L}^{\text{phys}} \;+\; \mathcal{L}^{\text{counter}}_{\text{terms}} \tag{S.14}$$

where

$$\mathcal{L}^{\text{phys}} \;=\; \tfrac{1}{2}(\partial\Phi_r)^2 \;-\; \tfrac{1}{2}m^2_{\text{phys}}\Phi_r^2 - \;\tfrac{1}{4!}\lambda_{\text{phys}}\Phi_r^4 + \;\overline{\Psi}_r(i\,\partial\!\!\!/ - M_{\text{phys}})\Psi_r \;-\; ig_{\text{phys}}\,\Phi_r\overline{\Psi}_r\gamma^5\Psi_r\,, \tag{S.15}$$

the counterterms are as in eq. (S.12) (where $\Phi\equiv\Phi_r$ and $\Psi\equiv\Psi_r$),

$$\begin{gathered}\delta^\phi_Z = Z_\phi - 1, \qquad \delta^\psi_Z = Z_\psi - 1, \qquad \delta^\phi_m = Z_\phi m_0^2 - m^2_{\text{phys}}\,, \qquad \delta^\psi_m = Z_\psi M_0 - M_{\text{phys}},\\ \delta_\lambda = Z_\phi^2\lambda_0 - \lambda_{\text{phys}}\,, \qquad \text{and} \quad \delta_g = Z_\psi Z_\phi^{1/2} g_0 - g_{\text{phys}}\,.\end{gathered}$$

We shall see momentarily that at the one-loop level of the theory we already need all the counterterms (S.12). In particular, we do need δ_λ even if we start with $\lambda_{\text{phys}}=0$. Thus, from the bare Lagrangian point of view, $\lambda_{\text{phys}}=0$ has no special meaning: $\lambda_0\neq 0$ and vanishing of a particular scattering amplitude we use to define the physical λ would be just an accident. In other words, we may *fine tune* λ_0 to achieve $\lambda=0$ just as we can fine tune λ_0 to achieve any other experimental value of the physical coupling, but it would not have any special meaning for the theory itself.

This is an example of the general rule: *barring fine tuning of the coupling parameters, a renormalizable quantum field theory has all the renormalizable couplings consistent with the theory's symmetries.* For the theory at hand, we have a Dirac field Ψ, a real pseudoscalar field Φ, and all the Lagrangian terms involving these fields should be invariant under Lorentz and parity transformations and have canonical dimensions ≤ 4 (for renormalizability's sake). There is only a finite number of such terms, and it is easy to see that the Lagrangian (S.15) comprises all such terms and no others. Consequently, the renormalized theory would not have any additional interactions.

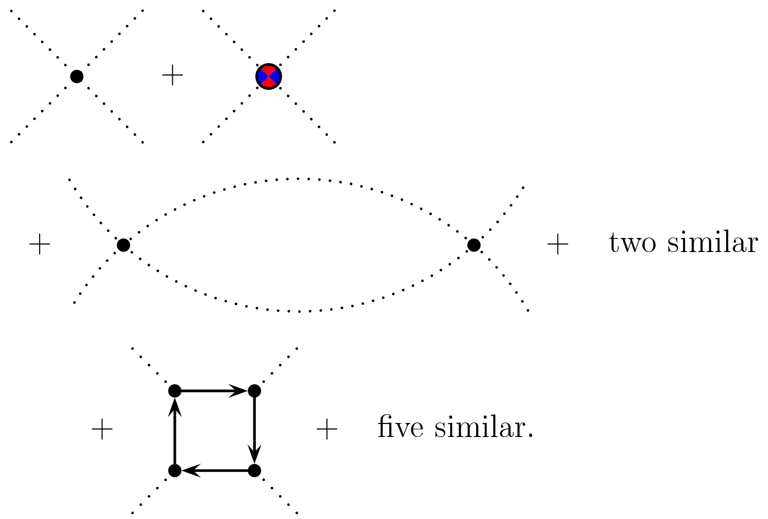
Sometimes, in absence of some coupling the theory has an additional symmetry that would not be present otherwise. In such case, the extra symmetry would prevent such coupling from being restored by the renormalization procedure. For example, consider the Lagrangian (S.15) for $g=0$ (but $\lambda\neq 0$): In the absence of the Yukawa coupling, the theory has an extra symmetry

$\Phi(x) \to -\Phi(x)$ (without parity), and this extra symmetry would prevent the renormalization procedure from restoring the Yukawa coupling. On the other hand, when $\lambda = 0$ but $g \neq 0$, the theory does not has any additional symmetries it wouldn't have for $\lambda \neq 0$, and that's why the renormalization gives rise to the $\lambda\Phi^4$ coupling even if it wasn't there to begin with.

Problem **2**(b):

At this stage we are ready to calculate the counterterms, beginning with the δ_λ. At the one-loop level of analysis, the four-boson amplitude comprises the following Feynman diagrams:

$$i\mathcal{M}^{1\,\text{loop}}(k_1,k_2,k_3,k_4) \;=$$



(S.16)

The last diagram here yields

$$-\int\frac{d^4p_1}{(2\pi)^4}\,\mathrm{Tr}\left\{(-g\gamma^5)\,\frac{i}{\not{p}_1 - M + i0}\,(-g\gamma^5)\,\frac{i}{\not{p}_2 - M + i0}\,(-g\gamma^5)\,\frac{i}{\not{p}_3 - M + i0}\,(-g\gamma^5)\,\frac{i}{\not{p}_4 - M + i0}\right\} \tag{S.17}$$

where

$$p_2 = p_1 + k_1\,, \qquad p_3 = p_2 + k_2, \qquad p_4 = p_3 + k_3 \quad\text{and}\quad p_1 = p_4 + k_4\,;$$

there are five similar diagrams related by permutations of the external momenta k_1, k_2, k_3, k_4. For generic values of these momenta, the integral (S.17) is quite complicated, but its divergence is k-independent and hence may be evaluated for any particular choice of k_i we find convenient. Clearly, the simplest set of k_i is $k_1 = k_2 = k_3 = k_4 = 0$; this is off-shell, but that's OK. Consequently, the integral (S.17) becomes

$$\begin{aligned} i\mathcal{M}^{\psi\,\rm loop}(0,0,0,0) \;&=\; -\int\!\frac{d^4p_1}{(2\pi)^4}\,{\rm Tr}\left((-g\gamma^5)\,\frac{i}{\not{p}-M+i0}\right)^4 \\ &=\; -g^4\int\!\frac{d^4p_1}{(2\pi)^4}\,\frac{{\rm tr}[\gamma^5(\not{p}+M)]^4}{(p^2-M^2+i0)^4} \qquad\qquad ({\rm S.18}) \\ &=\; \int\!\frac{d^4p_1}{(2\pi)^4}\,\frac{-4g^4}{(p^2-M^2+i0)^2} \end{aligned}$$

where the last equality follows from

$$[\gamma^5(\not{p}+M)]^2 \;=\; \gamma^5(\not{p}+M)\gamma^5(\not{p}+M) \;=\; (-\not{p}+M)(\not{p}+M) \;=\; -p^2\;+\;M^2 \qquad ({\rm S.19})$$

and hence ${\rm tr}[\gamma^5(\not{p}+M)]^4 = 4(p^2-M^2)^2$. Evaluating the integral on the last line of eq. (S.18) using dimensional regularization, we obtain

$$\mathcal{M}^{\psi\,\rm loop}(0,0,0,0) \;=\; \frac{-4g^4}{16\pi^2}\left(\frac{1}{\bar\epsilon}+\log\frac{\mu^2}{M^2}\right). \qquad ({\rm S.20})$$

It remains to multiply this result by 6 (for six similar diagrams) and add contributions of the other diagrams (S.16). The latter diagrams have been evaluated in class in the context of the scalar $\lambda\Phi^4$ theory, thus

$$\mathcal{M}(0,0,0,0) \;=\; -\lambda\;-\;\delta_\lambda\;+\;\frac{3\lambda^2}{32\pi^2}\left(\frac{1}{\bar\epsilon}+\log\frac{\mu^2}{m^2}\right)\;-\;\frac{24g^4}{16\pi^2}\left(\frac{1}{\bar\epsilon}+\log\frac{\mu^2}{M^2}\right)\;+\;O(\lambda^3,\;\lambda g^4\text{ or }g^6). \qquad ({\rm S.21})$$

The renormalization condition for the physical λ coupling is $\mathcal{M}=-\lambda$ when all external momenta are on shell and at the threshold ($s=4m^2$, $t=u=0$). At other values of external momenta, we should have

$$\mathcal{M}\;=\;-\lambda\;-\frac{\lambda^2}{32\pi^2}\times\text{finite}\;-\;\frac{4g^4}{16\pi^2}\times\text{finite}\;+\;\text{higher loop orders}; \qquad ({\rm S.22})$$

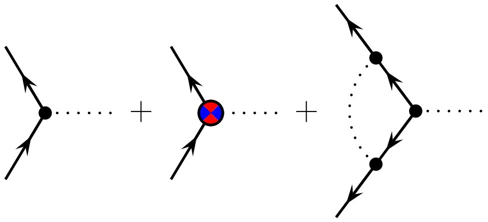
comparing this formula with eq. (S.21) finally gives us

$$\delta_\lambda^{1\,\rm loop}\;=\;\frac{3\lambda^2}{32\pi^2}\left(\frac{1}{\bar\epsilon}+\log\frac{\mu^2}{m^2}+\text{finite}\right)\;-\;\frac{24g^4}{16\pi^2}\left(\frac{1}{\bar\epsilon}+\log\frac{\mu^2}{M^2}+\text{finite}\right). \qquad ({\rm S.23})$$

As promised, $\delta_\lambda\neq 0$ even for $\lambda=0$ thanks to the renormalization due to fermionic loops.

Next, we want to calculate the δ_g counterterm, so let us consider the $\Phi\overline{\Psi}\gamma^5\Psi$ vertex correction. At the one-loop level of analysis,

$$-\Gamma^{(5)}(p',p) \;=\; \tag{S.24}$$



$$\begin{aligned} &= -g\gamma^5 \;-\; \delta_g\gamma^5 \\ &\quad + \int\!\frac{d^4k}{(2\pi)^4}\,(-g\gamma^5)\,\frac{i}{\not{p}'+\not{k}-M+i0}\,(-g\gamma^5)\,\frac{i}{\not{p}+\not{k}-M+i0}\,(-g\gamma^5)\times\frac{i}{k^2-m^2+i0}\,. \end{aligned}$$

As usual, we rewrite the integrand of the last integral as $ig^3\mathcal{N}^5/\mathcal{D}$ where

$$\begin{aligned} \frac{1}{\mathcal{D}} &= \frac{1}{k^2-m^2+i0}\times\frac{1}{((p+k)^2-M^2+i0}\times\frac{1}{(p'+k)^2-M^2+i0} \\ &= \iiint\! dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2}{(\ell^2-\Delta+i0)^3}\,, \\ \ell^2-\Delta &= x\big(k^2-m^2\big) \;+\; y\big((p+k)^2-M^2\big) \;+\; z\big((p'+k)^2-M^2\big) \;\Rightarrow \\ &\Rightarrow \begin{cases} \ell \;=\; k \;+\; xp \;+\; yp', \\ \Delta \;=\; zm^2 \;+\; (1-z)M^2 \;-\; xzp^2 \;-\; yzp'^2 \;-\; xyq^2 \\ \;=\; zm^2 \;+\; (1-z)^2M^2 \;-\; xyq^2 \quad \text{for on-shell } p^2=p'^2=M^2. \end{cases} \end{aligned} \tag{S.25}$$

As to the numerator $\mathcal{N}^5$, we have

$$\begin{aligned} \mathcal{N}^5 &= \gamma^5(\not{p}'+\not{k}+M)\gamma^5(\not{p}+\not{k}+M)\gamma^5 \;=\; (M-\not{k}-\not{p}')\gamma^5(M-\not{k}-\not{p}) \\ &= (M-\not{\ell}-z\,\not{p}'-x\,\not{q})\gamma^5(M-\not{\ell}-z\,\not{p}+y\,\not{q}) \end{aligned}$$

$\langle\!\langle$ in the context of $\bar{u}(p')\Gamma^{(5)}u(p)$ $\rangle\!\rangle$

$$\begin{aligned} &\cong (M-\not{\ell}-zM-x\,\not{q})\gamma^5(M-\not{\ell}-zM+y\,\not{q}) \\ &= \gamma^5\big((1-z)M+\not{\ell}+x\,\not{q}\big)\big((1-z)M-\not{\ell}+y\,\not{q}\big) \\ &\cong \gamma^5\left[-\ell^2 \;+\; (1-z)^2M^2 \;+\; xyq^2 \;+\; (1-z)^2M\,\not{q}\right] \\ &\cong \gamma^5\left[-\ell^2 \;-\; (1-z)^2M^2\right] \end{aligned} \tag{S.26}$$

where the last equivalence follows from

$$\bar{u}(p')\gamma^5 \not{q} u(p) \;=\; \bar{u}(p')(-\not{p}'\gamma^5 - \gamma^5 \not{p})u(p) \;=\; -2M\bar{u}(p')\gamma^5 u(p) \;\Longrightarrow\; \gamma^5 \not{q} \;\cong\; -2M\gamma^5. \tag{S.27}$$

Thus, the loop integral in eq. (S.24) becomes

$$-2ig^3\gamma^5 \int d(FP) \int \frac{d^4\ell}{(2\pi)^4}\, \frac{\ell^2 \,+\, (1-z)^2M^2}{(\ell^2-\Delta+i0)^3} \tag{S.28}$$

where we evaluate the momentum integral using Wick rotation and dimensional regularization

$$\begin{aligned}
-2i\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\ell^2\,+\,(1-z)^2M^2}{(\ell^2-\Delta+i0)^3} \;&\longrightarrow\; 2\mu^{4-D}\int\frac{d^D\ell_E}{(2\pi)^D}\,\frac{\ell_E^2\,-\,(1-z)^2M^2}{(\ell^2+\Delta)^3} \\
&= \int_0^\infty dt\, t^2 e^{-t\Delta}\left(-\frac{\partial}{\partial t}\,-\,(1-z)^2M^2\right)\int\frac{d^D\ell_E}{(2\pi)^D}\,\mu^{4-D}\,e^{-t\ell_E^2} \\
&= \frac{(4\pi\mu^2)^\epsilon}{16\pi^2}\int_0^\infty dt\, t^2 e^{-t\Delta}\left(+\tfrac{D}{2}\,t^{-(D/2)-1}\,-\,(1-z)^2M^2\,t^{-(D/2)}\right) \\
&= \frac{(4\pi\mu^2)^\epsilon}{16\pi^2}\left((2-\epsilon)\Gamma(\epsilon)\Delta^{-\epsilon}\,-\,(1-z)^2M^2\,\Gamma(1+\epsilon)\Delta^{-1-\epsilon}\right) \\
&\xrightarrow[\epsilon\to 0]{}\; \frac{1}{16\pi^2}\left(\frac{2}{\bar{\epsilon}}\,-\,1\,+\,2\log\frac{\mu^2}{\Delta}\,-\,\frac{(1-z)^2M^2}{\Delta}\right).
\end{aligned} \tag{S.29}$$

It remains to integrate over the Feynman parameters x, y and z, which yields

$$-\Gamma^{(5)}_{1\,\mathrm{loop}}(p',p) \;=\; -\gamma^5\times\left[g\,+\,\delta_g\,-\,\frac{g^3}{16\pi^2}\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{\Delta}\,+\,F\big(\tfrac{m^2}{M^2},\tfrac{q^2}{M^2}\big)\right)\right] \tag{S.30}$$

where

$$F\big(\tfrac{m^2}{M^2},\tfrac{q^2}{M^2}\big) \;=\; \iiint dx\,dy\,dz\,\delta(x+y+z-1)\left(2\log\frac{M^2}{\Delta(x,y,z)}\,-\,1\,-\,\frac{(1-z)^2M^2}{\Delta(x,y,z)}\right) \tag{S.31}$$

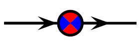
is a horribly complicated but finite function of the mass and momentum ratios.

In class, I have not explained the renormalization condition for the Yukawa coupling g, but it's clear that such condition should have form $\Gamma^{(5)} = g\gamma^5$ for the on-shell fermions and some particular value of the pseudoscalar's q^2, for example $q^2 = 0$ or on-shell $q^2 = m^2$ (allowed for $m \geq 2M$). Consequently, the Yukawa counterterm

$$\delta_g^{1\,\text{loop}} s \;=\; \frac{g^3}{16\pi^2}\left(\frac{1}{\bar\epsilon} + \log\frac{\mu^2}{\Delta} + F(\text{something specific})\right) \;\equiv\; \frac{g^3}{16\pi^2}\left(\frac{1}{\bar\epsilon} + \log\frac{\mu^2}{\Delta} + \text{finite}\right). \tag{S.32}$$

Our next targets are the fermion's mass and kinetic energy counterterms δ_M^ψ and δ_Z^ψ. At the one-loop level of analysis, the Dirac field's 1PI two-point Green's function is

$$-i\Sigma_\psi^{1\,\text{loop}}(\not p) \;=$$



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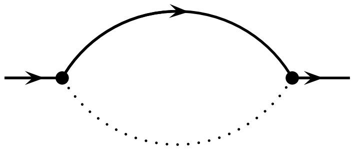


$$= \; -i\delta_M^\psi \;+\; i\delta_Z^\psi \not p \;+ \int\!\frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2 - m^2 + i0} \times (-g\gamma^5)\,\frac{i}{\not p + \not k - M + i0}\,(-g\gamma^5). \tag{S.33}$$

As usual, we write the loop integral as

$$-\,g^2 \int\!\frac{d^4k}{(2\pi)^4}\,\frac{\mathcal{N}}{\mathcal{D}} \tag{S.34}$$

where

$$\begin{aligned}
\frac{1}{\mathcal{D}} \;&=\; \frac{1}{k^2 - m^2 + i0} \times \frac{1}{(p+k)^2 - M^2 + i0} \;=\; \int_0^1\! dx\,\frac{1}{(\ell^2 - \Delta + i0)^2}\,, \\
\ell^2 \,-\, \Delta \;&=\; (1-x)[k^2 - m^2] \;+\; x[(k+p)^2 - M^2] \;\Rightarrow \\
&\Rightarrow \begin{cases} \ell \;=\; k \;+\; xp, \\ \Delta \;=\; (1-x)m^2 \;+\; xM^2 \;-\; x(1-x)p^2, \end{cases} \\
\mathcal{N} \;&=\; \gamma^5(\not p + \not k + M)\gamma^5 \;=\; M \,-\, \not p \,-\, \not k \\
&=\; M \;-\; (1-x)\not p \,-\, \not\ell \;\cong\; M \;-\; (1-x)\not p.
\end{aligned} \tag{S.35}$$

Therefore, the integral (S.34) becomes

$$-g^2 \int_0^1\! dx\,[M - (1-x)\not p]\int\!\frac{d^4k}{(2\pi)^4}\,\frac{1}{(\ell^2 - \Delta + i0)^2} \tag{S.36}$$

which we evaluate using dimensional regularization as

$$-i\frac{g^2}{16\pi^2}\int\limits_0^1 dx\,[M-(1-x)\,\not{p}\,]\,\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{\Delta}\right), \tag{S.37}$$

hence

$$\Sigma_{\psi}^{1\,\text{loop}}(\not{p})\;=\;\delta_M^{\psi}\,-\,\delta_Z^{\psi}\,\not{p}\,+\,\frac{g^2}{16\pi^2}\int\limits_0^1 dx\,[M-(1-x)\,\not{p}\,]\,\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{(1-x)m^2+xM^2-x(1-x)p^2}\right). \tag{S.38}$$

The renormalization conditions for the fermion's propagator correction $\Sigma^{\psi}(\not{p})$ are

$$\Sigma_{\psi}\Big|_{\not{p}=M}\;=\;0\qquad\text{and}\qquad\frac{\partial\Sigma_{\psi}}{\partial\,\not{p}}\bigg|_{\not{p}=M}\;=\;0. \tag{S.39}$$

In light of eq. (S.38), the second condition (S.39) becomes

$$\begin{aligned}\delta_Z^{\psi}[1\,\text{loop}]\;&=\;\frac{g^2}{16\pi^2}\,\frac{\partial}{\partial\,\not{p}}\int\limits_0^1 dx\,[M-(1-x)\,\not{p}\,]\,\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{(1-x)m^2+xM^2-x(1-x)p^2}\right)\bigg|_{\not{p}=M}\\&=\;\frac{g^2}{16\pi^2}\int\limits_0^1 dx\left[(x-1)\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{x^2M^2+(1-x)m^2}\right)\,+\,\frac{2x^2(1-x)M^2}{x^2M^2+(1-x)m^2}\right]\\&=\;-\frac{g^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{M^2}\,+\,\text{finite}\right).\end{aligned} \tag{S.40}$$

At the same time, the first condition (S.39) implies

$$\begin{aligned}&\delta_M^{\psi}[1\,\text{loop}]\;-\;M\delta_Z^{\psi}[1\,\text{loop}]\\&\qquad=\;-\frac{g^2}{16\pi^2}\int\limits_0^1 dx\,[M-(1-x)\,\not{p}\,]\,\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{(1-x)m^2+xM^2-x(1-x)p^2}\right)\bigg|_{\not{p}=M}\\&\qquad=\;-\frac{g^2}{16\pi^2}\int\limits_0^1 dx\,xM\times\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{x^2M^2+(1-x)m^2}\right)\\&\qquad=\;-\frac{g^2M}{32\pi^2}\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{M^2}\,+\,\text{finite}\right)\end{aligned} \tag{S.41}$$

and consequently

$$\delta^{\psi}_{M}[1\,\text{loop}] \;=\; -\frac{g^2 M}{16\pi^2}\left(\frac{1}{\bar{\epsilon}} \;+\; \log\frac{\mu^2}{M^2} \;+\; \text{finite}\right). \tag{S.42}$$

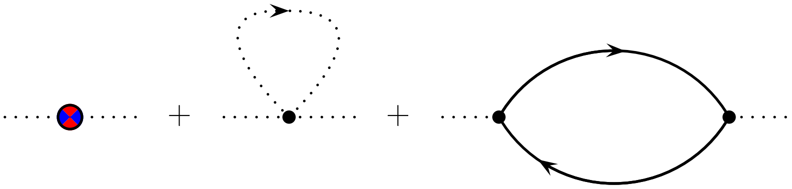
Note that similar to QED, the fermionic mass counterterm in the Yukawa theory is proportional to the mass itself and diverges logarithmically rather than linearly in the UV cutoff (*cf.* integral (S.36) prior to dimensional regularization). As in QED, this behavior is due an additional symmetry the Yukawa theory acquires when the fermion mass vanishes. Specifically, for $M = 0$ we have a *discrete chiral symmetry*

$$\Psi(x) \;\to\; \gamma^5\Psi(x), \qquad \Psi(x) \;\to\; -\Phi(x). \tag{S.43}$$

Unlike the gauge coupling in QED, the pseudoscalar Yukawa coupling does not respect continuous chiral transforms $\Psi(x) \to \exp(i\alpha\gamma^5)\Psi(x)$, but the discrete symmetry is sufficient for preventing the massless Yukawa theory from developing a mass shift via loop corrections.

Finally, consider the boson's mass and kinetic energy counterterms δ^{ϕ}_{M} and δ^{ϕ}_{Z}. At the one-loop level of analysis, the pseudoscalar field's 1PI two-point Green's function is

$$-i\Sigma^{1\,\text{loop}}_{\phi}(k^2) \;=$$



$$= \; -i\delta^{\phi}_{m} \;+\; i\delta^{\phi}_{Z}\,k^2 \;+\; \frac{i\lambda m^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} + 1 + \log\frac{\mu^2}{m^2}\right)$$
$$-\int\frac{d^4p}{(2\pi)^4}\,\text{Tr}\left(\frac{i}{\not{p} - M + i0}\,(-g\gamma^5)\,\frac{i}{\not{p} + \not{k} - M + i0}\,(-g\gamma^5)\right). \tag{S.44}$$

Again, we write the loop integral as

$$+\,g^2\int\frac{d^4p}{(2\pi)^4}\,\frac{\mathcal{N}}{\mathcal{D}} \tag{S.45}$$

where

$$\begin{aligned}
\frac{1}{\mathcal{D}} &= \frac{1}{p^2-M^2+i0}\times\frac{1}{(p+k)^2-M^2+i0} = \int\limits_0^1 dx\frac{1}{(\ell^2-\Delta+i0)^2}\,,\\
\ell^2\,-\,\Delta &= (1-x)[p^2-M^2]\;+\;x[(p+k)^2-M^2]\;\Rightarrow\\
\Rightarrow &\begin{cases} \ell\;=\;p\;+\;xk,\\ \Delta\;=\;M^2\;-\;x(1-x)k^2,\end{cases}\\
\mathcal{N} &= \mathrm{Tr}\big[(\not{p}+M)\gamma^5(\not{p}+\not{k}+M)\gamma^5\big] \;=\; \mathrm{Tr}\big[(M+\not{p})(M-\not{p}-\not{k})\big]\\
&= 4M^2\;-\;4p(p+k)\;=\;4M^2\;-\;4(\ell-xk)(\ell+k-xk)\\
&\cong 4M^2\;-\;4\ell^2\;+\;4x(1-x)k^2\;=\;8M^2\;-4\Delta\;-\;4\ell^2.
\end{aligned}\tag{S.46}$$

Consequently, the integral (S.45) becomes

$$4g^2\int\limits_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\,\frac{2M^2-\Delta-\ell^2}{(\ell^2-\Delta+i0)^2}\tag{S.47}$$

where we evaluate the $\int d^4\ell$ using the same techniques as in eq. (S.29):

$$\begin{aligned}
\int\frac{d^4\ell}{(2\pi)^4}\,\frac{2M^2-\Delta-\ell^2}{(\ell^2-\Delta+i0)^2} &\;\longrightarrow\; i\mu^{4-D}\int\frac{d^D\ell_E}{(2\pi)^D}\,\frac{2M^2-\Delta+\ell_E^2}{(\ell_E^2+\Delta)^2}\\
&= i\mu^{4-D}\int\limits_0^\infty dt\,te^{-t\Delta}\left(2M^2-\Delta-\frac{\partial}{\partial t}\right)\int\frac{d^D\ell_E}{(2\pi)^D}\,e^{-t\ell_E^2}\\
&= \frac{i\mu^{4-D}}{(4\pi)^{D/2}}\int\limits_0^\infty dt\,te^{-t\Delta}s\left((2M^2-\Delta)t^{-(D/2)}\;+\;\tfrac{D}{2}\,t^{-(D/2)-1}\right)\\
&= \frac{i\mu^{4-D}}{(4\pi)^{D/2}}\left((2M^2-\Delta)\Gamma(2-\tfrac{D}{2})\Delta^{(D/2)-2}\;+\;\tfrac{D}{2}\Gamma(1-\tfrac{D}{2})\Delta^{(D/2)-1}\right)\\
&= \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon}\left(2M^2\,-\,\Delta\,-\,\frac{2-\epsilon}{1-\epsilon}\Delta\right)\\
&\xrightarrow[\epsilon\to0]{}\;\frac{i}{16\pi^2}\left[(2M^2-3\Delta)\left(\frac{1}{\bar{\epsilon}}\,+\,\log\frac{\mu^2}{\Delta}\right)\;-\;\Delta\right].
\end{aligned}\tag{S.48}$$

Consequently,

$$\begin{aligned}\Sigma_\phi^{1\,\mathrm{loop}}(k^2)\ &=\ \delta_m^\phi\ -\ \delta_Z^\phi\,k^2\ -\ \frac{\lambda m^2}{32\pi^2}\left(\frac{1}{\bar\epsilon}\,+\,1\,+\,\log\frac{\mu^2}{m^2}\right)\\ &\qquad -\ \frac{g^2}{4\pi^2}\int\limits_0^1\!dx\left[(2M^2-3\Delta)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{\Delta}\right)\ -\ \Delta\right].\end{aligned}\tag{S.49}$$

Similarly to the fermion's propagator correction Σ_ψ discussed above, the renormalization conditions for a scalar or a pseudoscalar field are

$$\Sigma_\phi\big|_{k^2=m^2}\ =\ 0\qquad\text{and}\qquad\frac{\partial\Sigma_\phi}{\partial k^2}\bigg|_{k^2=m^2}\ =\ 0.\tag{S.50}$$

Therefore, in light of eq. (S.49),

$$\begin{aligned}\delta_Z^\phi[1\,\mathrm{loop}]\ &=\ -\frac{g^2}{4\pi^2}\frac{\partial}{\partial k^2}\int\limits_0^1\!dx\bigg[\big(3x(1-x)k^2-3M^2\big)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{M^2-x(1-x)k^2}\right)\\ &\qquad\qquad\qquad\qquad -\ \big(M^2-x(1-x)k^2\big)\bigg]_{k^2=m^2}\\ &=\ -\frac{g^2}{4\pi^2}\int\limits_0^1\!dx\,x(1-x)\left[\frac{3}{\bar\epsilon}\ +\ 3\log\frac{\mu^2}{M^2-x(1-x)m^2}\ +\ \frac{2x(1-x)m^2}{M^2-x(1-x)m^2}\right]\\ &=\ -\frac{g^2}{8\pi^2}\left(\frac{1}{\bar\epsilon}\ +\ \log\frac{\mu^2}{M^2}\ +\ \text{finite}\right).\end{aligned}\tag{S.51}$$

Likewise,

$$\begin{aligned}&\delta_m^\phi[1\,\mathrm{loop}]\ -\ m^2\,\delta_Z^\phi[1\,\mathrm{loop}]\ -\ \frac{\lambda m^2}{32\pi^2}\left(\frac{1}{\bar\epsilon}\,+\,1\,+\,\log\frac{\mu^2}{m^2}\right)\\ &\qquad=\ -\frac{g^2}{4\pi^2}\int\limits_0^1\!dx\bigg[\big(3x(1-x)k^2-3M^2\big)\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{M^2-x(1-x)k^2}\right)\\ &\qquad\qquad\qquad\qquad -\ \big(M^2-x(1-x)k^2\big)\bigg]_{k^2=m^2}\\ &\qquad=\ -\frac{g^2}{4\pi^2}\left[(\tfrac{1}{2}m^2-M^2)\times\left(\frac{1}{\bar\epsilon}\,+\,\log\frac{\mu^2}{M^2}\right)\ +\ \text{finite}\right]\end{aligned}\tag{S.52}$$

and hence

$$\delta_m^{\phi}[1\,\text{loop}] \;=\; \left[\frac{\lambda m^2}{32\pi^2} \;+\; \frac{g^2(M^2-m^2)}{8\pi^2}\right] \times \left(\frac{1}{\bar{\epsilon}} \,+\, \log\frac{\mu^2}{M^2}\right) \;+\; \text{finite}. \tag{S.53}$$