Problem **9.2**(a):

First, consider the partition function analytically continued to an imaginary temperature $\mathcal{T} = 1/(iT)$, *i.e.* $\beta = iT$. Formally,

$$\begin{aligned} Z(1/iT) &= \operatorname{Tr}\left[e^{-iT\hat{H}} = \hat{\mathcal{U}}(T,0)\right] = \int dx_0\, U(x_0,T;x_0,0) \\ &= \int dx_0 \iint\limits_{x(0)=x_0}^{x(T)=x_0} \mathcal{D}'[x(t)]\, e^{iS[x(t)]} \\ &= \iint\limits^{x(T)=x(0)} \mathcal{D}[x(t)]\, e^{iS[x(t)]} \end{aligned} \tag{S.1}$$

where $S$ is the Minkowski-time Lagrangian action functional

$$S[x(t)] = \int_0^T dt\left(\tfrac{1}{2}\dot{x}^2 - V(x)\right) : \tag{S.2}$$

for simplicity, we work in units $\hbar = 1, m = 1$. Note the boundary conditions in the last path integral in eq. (S.1): $x(t)$ is required to be periodic in time, $x(T) = x(0)$ but there are no separate initial or final conditions.

Strictly speaking, the Minkowski-time path integral is defined as an analytic continuation of the Euclidean-time path integral over $x(t_E)$, $t_E = it_{\text{Mink}}$. The boundary conditions should be analytically continued too — which for the integral (S.1) means $x(t_E = iT) = x(t_E = 0)$. In other words, we should analytically continue *back* to the real inverse temperature $\beta = 1/\mathcal{T}$, thus

$$Z(\mathcal{T}) = \iint\limits^{x(\beta)=x(0)} \mathcal{D}[x(t_E)]\, e^{-S_E[x(t_E)]} \tag{S.3}$$

where the Euclidean-time action is

$$S_E[x(t_E)] = \int_0^\beta dt_E\left(\tfrac{1}{2}\dot{x}^2 + V(x)\right). \tag{S.4}$$

Problem **9.2**(b):

Strictly speaking, we should first discretize the Euclidean path integral and only then Fourier transform the discrete Euclidean time. Thus,

$$t_E \;\to\; t_n = \frac{n\beta}{N}, \quad n = 0,1,2,\ldots,N, \quad t_0 \equiv t_N,$$
$$x(t_E) \;\to\; x_n = x(t_n), \quad x_0 \equiv x_N,$$

the path integral is

$$Z \;=\; lim(N\to\infty)\, C(N) \int d^N x\, e^{-S_E^{\rm discr}(x_1,\ldots,x_N)}, \tag{S.5}$$

where

$$C(N) \;=\; \left(\frac{N}{2\pi\beta}\right)^{N/2} \tag{S.6}$$

is the Euclidean normalization factor and the Discretized Euclidean Action for the harmonic oscillator is

$$S_E^{\rm discr}(x_1,\ldots,x_N) \;=\; \frac{N}{2\beta}\sum_{n=1}^{N}(x_n - x_{n-1})^2 + \frac{\omega^2\beta}{2N}\sum_{n=1}^{N} x_n^2\,. \tag{S.7}$$

The action (S.7) is quadratic with respect to the integration variables $x_1,\ldots,x_N$, so the integral (S.5) is Gaussian and may be evaluated exactly. Unfortunately, the determinant of the quadratic form (S.7) is rather formidable, so let us first diagonalize the action. The continuum-time Euclidean action is diagonalized via Fourier transform

$$\begin{aligned} x(t_E) \;&=\; \sum_{k=-\infty}^{+\infty} \beta^{-1/2}\, e^{-2\pi i k t_e/\beta} y_k\,, \\ S_E[x] \;&=\; \tfrac{1}{2}\sum_k \left(\omega^2 \;+\; \frac{(2\pi k)^2}{\beta^2}\right)\, |y_k|^2 : \end{aligned} \tag{S.8}$$

note that the frequencies here are discrete because the Euclidean time is periodic; also, $y_k^* = y_{-k}$. For the discretized action (S.7) however, we need the discrete Fourier transform

$$x_n \;=\; \frac{1}{\sqrt{N}}\sum_{k=1}^{N} e^{-2\pi i k n/N} y_k \tag{S.9}$$

where the discrete frequencies $k$ are defined modulo $N$, *i.e.* $y_0 \equiv y_N$, $y_{-k} \equiv y_{N-k}$, *etc.*, *etc.*;

again, the frequency modes $y_k$ are complex, but the complete set of $y_1, \dots y_N$ is self-conjugate as $y_k^* = y_{-k}$. The key formula of the discrete Fourier transform is

$$\sum_n e^{-2\pi i(k-\ell)n/N} \;=\; N\delta^{\mathrm{mod}\,N}(k-\ell). \tag{S.10}$$

Consequently,

$$\sum_n^{\mathrm{mod}\,N} x_n^2 \;=\; \sum_n^{\mathrm{mod}\,N} x_n^* x_n \;=\; \sum_k^{\mathrm{mod}\,N} y_k^* y_k \tag{S.11}$$

and likewise

$$\sum_n^{\mathrm{mod}\,N} (x_n - x_{n-1})^2 \;=\; \sum_k^{\mathrm{mod}\,N} \left|1 - e^{2\pi ik/N}\right|^2 \; |y_k|^2 \tag{S.12}$$

where the latter follows from

$$x_n - x_{n-1} \;=\; \sum_k^{\mathrm{mod}\,N} N^{-1/2} e^{-2\pi ikn/N} \left(1 - e^{2\pi ik/N}\right) y_k \,.$$

Hence,

$$S_E^{\mathrm{discr}}[y_k] \;=\; \tfrac{1}{2} \sum_k^{\mathrm{mod}\,N} \left(\frac{4N}{\beta}\,\sin^2\frac{\pi k}{N} \;+\; \frac{\omega^2\beta}{N}\right) |y_k|^2 \tag{S.13}$$

and therefore

$$\begin{aligned} Z \;&=\; \lim_{N\to\infty} C(N)J(N)\int d^N y\, e^{-S_E^{\mathrm{discr}}(y)} \\ &=\; \lim_{N\to\infty} C(N)J(N) \prod_k^{\mathrm{mod}\,N} \left(\frac{4N}{\beta}\,\sin^2\frac{\pi k}{N} \;+\; \frac{\omega^2\beta}{N}\right)^{1/2} \end{aligned} \tag{S.14}$$

where $J(N)$ is the Jacobian of the discrete Fourier transform (S.9). To evaluate this Jacobian, we perform the Fourier transform twice:

$$y_k \;=\; \sum_m^{\mathrm{mod}\,N} N^{-1/2} e^{-2\pi imk/N} z_m, \qquad x_n \;=\; \sum_k^{\mathrm{mod}\,N} N^{-1/2} e^{-2\pi ikn/N} y_k \;=\; (-1)^n z_n\,, \tag{S.15}$$

which immediately tells us that

$$\left[\det\frac{\partial x_n}{\partial y_k}\right]^2 \;=\; \det\frac{\partial x_n}{\partial z_m} \;=\; \pm 1$$

and hence $J = |\det(\partial x_n/\partial y_k)| = 1$ and therefore

$$Z(\beta,\omega,N) \;=\; \prod_{k}^{\text{mod}\,N}\left(4\sin^2\frac{\pi k}{N} \,+\, \frac{\omega^2\beta^2}{N^2}\right)^{-1/2}. \tag{S.16}$$

At this point, let me use without proof a somewhat obscure mathematical formula

$$\prod_{k=1}^{N-1}\left(2\sin\frac{\pi k}{N}\right) \;=\; N, \tag{S.17}$$

which allows me to re-write the discretized partition function as

$$\begin{aligned} Z(\beta,\omega,N) \;&=\; \frac{N}{\omega\beta}\times\prod_{k=1}^{N-1}\left(4\sin^2\frac{\pi k}{N} \,+\, \frac{\omega^2\beta^2}{N^2}\right)^{-1/2} \\ &=\; \frac{1}{\omega\beta}\times\prod_{k=1}^{N-1}\left(1 \,+\, \frac{\omega^2\beta^2}{4N^2\sin^2\frac{\pi k}{N}}\right)^{-1/2}. \end{aligned} \tag{S.18}$$

To evaluate the large $N$ limit of this partition function (physically, the continuous time limit), we approximate $4N^2\sin^2(\pi k/N) \approx (2\pi k)^2$ for $k \ll N$, and likewise $4N^2\sin^2(\pi k/N) \approx (2\pi(N-k))^2$ for $(N-k) \ll N$ while for the remaining modes $\sin^2(\pi k/N) = O(1)$ and hence

$$1 \,+\, \frac{\omega^2\beta^2}{4N^2\sin^2\frac{\pi k}{N}} \;\approx 1.$$

Consequently,

$$\begin{aligned} Z(\beta,\omega,N) \;&\xrightarrow[N\gg 1]{}\; \frac{1}{\omega\beta}\times\prod_{1\le k\ll N}\left(1 \,+\, \frac{\omega^2\beta^2}{(2\pi k)^2}\right)^{-1/2} \times \prod_{1\le(N-k)\ll N}\left(1 \,+\, \frac{\omega^2\beta^2}{(2\pi(N-k))^2}\right)^{-1/2} \\ &\xrightarrow[N\to\infty]{}\; \frac{1}{\omega\beta}\times\prod_{k=1}^{\infty}\left(1 \,+\, \frac{\omega^2\beta^2}{(2\pi k)^2}\right)^{-1}. \end{aligned} \tag{S.19}$$

The above calculation was rigorous but rather long. The intent of this exercise was for you to derive eq. (S.19) using continuous rather than discrete space time. This means, diagonalizing the continuous-time Euclidean Action via the Fourier transform (S.8) using $\mathcal{D}[x(t_E)] = \mathcal{D}[y_k] \times$ a constant Jacobian (possibly dependent on the time interval $\beta$ but not on the $\omega$) and

$$\int\!\!\mathcal{D}[y_k] \;=\; \prod_{k=-\infty}^{+\infty}\int dy_k \;=\int dy_0 \times \prod_{k=1}^{+\infty}\iint d\,\mathrm{Re}\,y_k\, d\,\mathrm{Im}\,y_k\,. \tag{S.20}$$

Thus,

$$\begin{aligned} Z \;&\propto \int\!\!\mathcal{D}[y_k]\exp\!\left(-\tfrac{1}{2}\sum_{k=-\infty}^{+\infty}\left(\omega^2+\frac{(2\pi k)^2}{\beta^2}\right)|y_k|^2\right) \\ &= \sqrt{\frac{2\pi}{\omega^2}}\times\prod_{k=1}^{+\infty}\left(\frac{\pi}{\omega^2+\frac{(2\pi k)^2}{\beta^2}}\right) \\ &= \frac{\sqrt{2\pi}}{\omega}\times\prod_{k=1}^{+\infty}\left(\frac{\beta^2}{4\pi k^2}\times\frac{1}{1+\frac{\omega^2\beta^2}{(2\pi k)^2}}\right) \\ &\propto \frac{1}{\omega}\times\prod_{k=1}^{+\infty}\left(\frac{1}{1+\frac{\omega^2\beta^2}{(2\pi k)^2}}\right) \end{aligned} \tag{S.21}$$

In other words, the partition function is

$$Z(\beta,\omega) \;=\; \frac{A(\beta)}{\omega}\times\prod_{k=1}^{+\infty}\left(\frac{1}{1+\frac{\omega^2\beta^2}{(2\pi k)^2}}\right)^{-1} \tag{S.22}$$

where $A(\beta)$ is some unknown overall coefficient, possibly divergent and / or $\beta$ dependent, but it cannot depend on the harmonic oscillator's frequency $\omega$. Indeed, the frequency dependence of the partition function (S.22) is in complete agreement with with the formula (S.19) we obtained above by discretizing the Euclidean time. The discretizing approach also yields fixes the overall coefficient $A = 1/\beta$; you decide whether it was worth the bother.

It remains to evaluate the infinite product in eq. (S.19). Consider $Z(\omega\beta)$ as an analytic function of a complex argument. Whenever any factor of on the right hand side has a zero in the complex $(\omega\beta)$ plane, $Z(\omega\beta)$ has a zero and ditto for the poles. Also, the product converges, so these are the only poles and zeroes of the $Z(\omega\beta)$ The individual factors at hand are $1/(\omega\beta)$

and

$$\frac{1}{1\,+\,\frac{\omega^2\beta^2}{(2\pi k)^2}}\ =\ \frac{(2\pi k)^2}{(\omega\beta+2\pi ki)\times(\omega\beta-2\pi ki)}$$

for $k=1,2,3,\ldots$. Thus, the $Z(\omega\beta)$ function has no zeroes and it has poles at $\omega\beta=2\pi ki$ for all integers $k$ (positive, negative and zero). In other words, it has the same poles and zeroes as the $1/\sinh(\omega\beta/2)$ function and indeed, there is a well known formula

$$\sinh(z)\ =\ z\prod_{k=1}^{+\infty}\left(1+\frac{z^2}{(\pi k)^2}\right)$$

(*cf.* the text of the problem), which immediately leads us to

$$Z(\omega,\beta)\ =\ \frac{1}{2\sinh(\omega\beta/2)} \tag{S.23}$$

Finally, let us compare our result (S.23) to the well-known Planck formula

$$Z\ =\sum_n\exp\bigl(-\beta E_n(\omega)\bigr)\ =\sum_{n=0}^{+\infty}\exp\bigl(-\beta\omega(n+\tfrac{1}{2})\bigr)\ =\ \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}\ =\ \frac{1}{2\sinh(\beta\omega/2)}\,, \tag{S.24}$$

which indeed agrees with the our path-integral-based formula (S.23), provided we have the correct overall coefficient.

Problem **9.2**(c):

Generalizing eq. (S.3) from particle mechanics to field theory is quite straightforward. For a real scalar field field $\phi(x)$ with a Euclidean Lagrangian

$$\mathcal{L}_E\ =\ \tfrac{1}{2}(\partial\phi)^2\ +\ V(\phi) \tag{S.25}$$

we have finite-temperature Partition function

$$Z(\beta)\ =\ \iint\limits_{\phi(\mathbf{x},\beta)=\phi(\mathbf{x},0)}\mathcal{D}[\phi(\mathbf{x},x_4)]\ \exp\left[-\int\!d^3\mathbf{x}\int_0^\beta\!dx_x\ \bigl(\tfrac{1}{2}(\partial\phi)^2\ +\ V(\phi)\bigr)\right]. \tag{S.26}$$

In other words, finite temperature translates into geometry of the Euclidean 4D spacetime: The Euclidean time $x_4=it$ is of finite extent $\beta=1/\mathcal{T}$ and the scalar field is subject to the periodic boundary condition; the other 3 dimensions $x_1,x_2,x_3$ are infinite as usual.

For the free scalar field, the Euclidean action is a quadratic functional

$$S_E[\phi(x_E)] \;=\; \tfrac{1}{2}\int d^4x_E\,\phi(m^2-\partial^2)\phi\,, \tag{S.27}$$

which becomes diagonal after a Fourier transform. However, because of the periodicity of the Euclidean time coordinate, the Euclidean "energies" $k_4$ have discrete rather than continuous spectrum,

$$\phi(x_E) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{k_4}\beta^{-1/2}e^{ik_Ex_E}\Phi(k_E) \tag{S.28}$$

where

$$k_4 \;=\; \frac{2\pi}{\beta}\times\text{integer}. \tag{S.29}$$

Consequently,

$$S_E \;=\; \tfrac{1}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{k_4}(m^2+k_E^2)|\Phi(k_E)|^2 \tag{S.30}$$

and hence

$$Z \;=\; \left[\mathrm{Det}(m^2-\partial_E^2)_{\text{periodic}}\right]^{-1/2} \;=\; \exp\left[\text{const}-\tfrac{1}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{k_4}\log(m^2+k^2)\right]. \tag{S.31}$$

It is often convenient to re-express a sum over a discrete momentum component using Poisson's re-summation formula:

$$\begin{aligned}\sum_{n=-\infty}^{+\infty}F(n) \;&=\; \int_{-\infty}^{+\infty}dx\,F(x)\times\sum_{n=-\infty}^{+\infty}\delta(x-n)\\ &=\; \int_{-\infty}^{+\infty}dx\,F(x)\times\sum_{\ell=-\infty}^{+\infty}e^{2\pi i\ell x}\end{aligned} \tag{S.32}$$

or for the problem at hand

$$\sum_{k_4}F(k_4) \;=\; \beta\sum_{\ell=-\infty}^{+\infty}\int\frac{dk_4}{2\pi}\,F(k_4)e^{i\beta\ell k_4}. \tag{S.33}$$

Hence, the Helmholtz free energy $\mathcal{F}=-\mathcal{T}\log Z$ of the free Hermitian scalar field can be written as

$$\mathcal{F} \;=\; \text{const} \;+\; \tfrac{1}{2} \sum_{\ell=-\infty}^{+\infty} \int \frac{d^4 k_E}{(2\pi)^4}\, e^{i\beta\ell k_4}\, \log(k_E^2 + m^2). \tag{S.34}$$

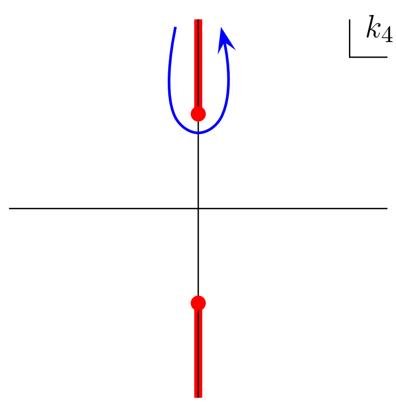
In the zero-temperature limit $\beta \to \infty$, the sum $\sum_\ell$ reduces to the $\ell = 0$ term while all the other terms are suppressed by the rapidly changing phase $e^{i\beta\ell k_4}$. In the general spirit of subtracting the zero-point energy contribution, we should therefore get rid of the $\ell = 0$ term. Since all the other terms come in symmetric pairs $\pm\ell \neq 0$, we arrive at

$$\mathcal{F} \;=\; \sum_{\ell=1}^{\infty} \int \frac{d^4 k_E}{(2\pi)^4}\, e^{i\beta\ell k_4}\, \log(k_E^2 + m^2). \tag{S.35}$$

Formula (S.35) has a nice 4D form, but for the purpose of comparison with the ordinary statistical mechanics, let us integrate over the $k_4$ before we integrate over the 3–momentum $\mathbf{k}$. For fixed $\mathbf{k}$ and $\ell$ we are faced with the integral

$$I \;=\; \int \frac{dk_4}{2\pi}\, e^{i\beta\ell k_4}\, \log(k_4^2 + E^2) \tag{S.36}$$

where $e^2 = m^2 + \mathbf{k}^2$. The logarithm here has branch cuts (in the complex $k_4$ plane) from $+iE$ to $+i\infty$ and also from $-iE$ to $-i\infty$, so we would like to deform the integration contour away from the real axis to make it wrap around the upper branch cut:




In other words, $k_4 = iE(1{+}x{+}i\epsilon)$ on its way down from $x = +\infty$ to $x = 0$ and $k_4 = iE(1{+}x{-}i\epsilon)$

on its way up from $x = 0$ back to $k = +\infty$, hence

$$\begin{aligned} I \;&=\; \frac{iE}{2\pi}\int\limits_0^{+\infty} dx\, e^{-\beta\ell E(1+x)} \times \left[\log(E^2(-2x - x^2 + i\epsilon)) \,-\, \log(E^2(-2x - x^2 - i\epsilon)) \;=\; 2\pi i\right] \\ &=\; -E\int\limits_0^{+\infty} dx\, e^{-\beta\ell E(1+x)} \;=\; -\frac{e^{-\beta\ell E}}{\beta\ell}\,. \end{aligned} \tag{S.37}$$

Next, we sum this integral over $\ell$, which gives gives

$$\frac{-1}{\beta}\sum_{\ell=1}^{\infty}\frac{(e^{-\beta E})^{\ell}}{\ell} \;=\; \mathcal{T}\log\left(1 \,-\, e^{-\beta E}\right) \tag{S.38}$$

and therefore free energy

$$\mathcal{F}(\mathcal{T}, m) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\mathcal{T}\log\left(1 \,-\, e^{-\beta E_{\mathbf{k}}}\right) \tag{S.39}$$

Finally, let us compare our result (S.39) with the conventional statistical mechanics of identical spinless relativistic bosons. In SM of identical bosons,

$$\mathcal{F}(\mathcal{T}, m) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,\mathcal{F}^{\text{harmonic}}_{\text{oscillator}}(\mathcal{T}, E_{\mathbf{k}}) \tag{S.40}$$

where each oscillator mode contributes

$$\mathcal{F}^{\text{harmonic}}_{\text{oscillator}}(\mathcal{T}, E_{\mathbf{k}}) \;=\; -\mathcal{T}\log Z^{\text{harmonic}}_{\text{oscillator}} \;=\; \mathcal{T}\log(2\sinh(E\beta/2)) \;=\; \tfrac{1}{2}E \,+\, \mathcal{T}\log\left(1 \,-\, e^{-\beta E}\right) \tag{S.41}$$

Subtracting the zero-point energy $\frac{1}{2}E$ and substituting into eq. (S.40) we arrive at precisely eq. (S.39), which shows that the Functional Quantization of the field theory correctly reproduces the free energy of the field's quanta.

Problem **9.2**(d):

For the $(0 + 1)$ dimensional (zero space, one time) free complex Grassmann field $\psi(t)$ we have quadratic Euclidean action

$$S_E \;=\; \int_0^\beta dt_E\, \bar{\psi}(\partial + \omega)\psi \tag{S.42}$$

and hence Partition Function

$$Z \;=\; \mathrm{Det}[\partial + \omega]\,. \tag{S.43}$$

All physical observables of this system must be periodic in Euclidean time, so the odd Grassmannians such as the fermionic fields themselves should be either periodic or antiperiodic. Therefore, the 'momentum' modes should be quantized as either integers or half integers,

$$k \;=\; \frac{2\pi}{\beta} \times n \qquad \text{or} \quad k \;=\; \frac{2\pi}{\beta} \times (n + \tfrac{1}{2}), \tag{S.44}$$

which produces two distinct expressions for the partition function: In the periodic case, we have

$$\begin{aligned} Z_+ \;&\propto\; \prod_k (ik + \omega) \\ &=\; \omega \times \prod_{n=1}^{\infty} \left( \omega^2 + \left( \frac{2\pi n}{\beta} \right)^2 \right) \\ &\propto\; \beta\omega \times \prod_{n=1}^{\infty} \left( 1 \,+\, \left( \frac{\beta\omega}{2\pi n} \right)^2 \right) \\ &=\; 2\sinh(\beta\omega/2), \end{aligned} \tag{S.45}$$

while the anti-periodic partition function is

$$\begin{aligned} Z_- \;&\propto\; \prod_k (ik + \omega) \\ &=\; \prod_{n=0}^{\infty} \left( \omega^2 + \left( \frac{2\pi}{\beta} \right)^2 (n + \tfrac{1}{2})^2 \right) \\ &\propto\; \prod_{n=0}^{\infty} \left( 1 \,+\, \left( \frac{\beta\omega}{\pi(n + \frac{1}{2}} \right)^2 \right) \\ &=\; \prod_{m=1}^{\infty} \left( 1 \,+\, \left( \frac{\beta\omega}{\pi m} \right)^2 \right) \bigg/ \prod_{m=1}^{\infty} \left( 1 \,+\, \left( \frac{\beta\omega}{2\pi m} \right)^2 \right) \\ &=\; \sinh(\beta\omega) \Big/ \sinh(\beta\omega/2) \;=\; 2\cosh(\beta\omega/2). \end{aligned} \tag{S.46}$$

In other words,

$$Z_{\pm}(\beta,\omega) \;=\; e^{+\beta\omega/2}\left(1 \mp e^{-\beta\omega}\right). \tag{S.47}$$

Physically, the anti-periodic partition function $Z_-$ agrees with the two-level Fermi statistics while the periodic function $Z_+$ does not seem to be a partition function of anything.

Therefore, *at finite temperature, the fermionic fields are anti-periodic in the Euclidean time.*

Naturally, the same rule applies to the fermionic fields in any space dimension. For example, for a free Dirac field in $d = (3+1)$, we have

$$Z \;=\; \left[\mathrm{Det}(m^2 - \partial_E^2)_{\mathrm{antiperiodic}}\right]^{+2} \;=\; \mathrm{const} \times \exp\left[+2\int\frac{d^3\mathbf{x}}{(2\pi)^3}\sum_{k_4}\log(k_E^2 + m^2)\right] \tag{S.48}$$

where the $k_4$ "energies" have the half-integral rather than integral spectrum. Consequently, the Poisson re-summation becomes

$$\sum_{k_4} F(k_4) \;=\; \beta\sum_{\ell=-\infty}^{+\infty}\int\frac{dk_4}{2\pi}\,F(k_4)\times e^{i\beta\ell k_4}\times(-1)^{\ell}, \tag{S.49}$$

which leads to the free energy

$$\mathcal{F} \;=\; \mathrm{const} \;-\; 2\sum_{\ell=-\infty}^{+\infty}(-1)^{\ell}\int\frac{d^4k_E}{(2\pi)^4}\,e^{i\beta\ell k_4}\,\log(k_E^2 + m^2) \tag{S.50}$$

or, after subtracting the zero-point energy

$$\mathcal{F} \;=\; 4\sum_{\ell=1}^{\infty}(-1)^{\ell-1}\int\frac{d^4k_E}{(2\pi)^4}\,e^{i\beta\ell k_4}\,\log(k_E^2 + m^2) \tag{S.51}$$

Similar to the bosonic case, we may re-write this formula in the conventional 3D terms by integration over the $k_4$ and summing over the $\ell$'s. The integration over the $k_4$ works exactly as in the bosonic case, but the sum $\sum_\ell$ is slightly different because of the alternating signs:

$$\sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\beta\ell}\, e^{-\beta E\ell} \;=\; -\mathcal{T}\,\log\Bigl(1\,+\,e^{-\beta E}\Bigr) \tag{S.52}$$

and hence

$$\mathcal{F}(\mathcal{T},m) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\,(-4\mathcal{T})\,\log\Bigl(1\,+\,e^{-\beta E_{\mathbf{k}}}\Bigr) \tag{S.53}$$

in full agreement with the Fermi–Dirac statistical mechanics.

Problem **9.2**(e):

Similarly to other bosonic fields, at finite temperature $\mathcal{T}=1/\beta$, the EM field $A^\mu(x_E)$ becomes periodic in the Euclidean time direction,

$$A^\mu(\mathbf{x},x_4=0) \;=\; A^\mu(\mathbf{x},x_4=\beta),\qquad \mu=1,2,3,4. \tag{S.54}$$

Its *local* properties however remain exactly the same; in particular, we still have local gauge transformations

$$A'^\mu(x_E) \;=\; A^\mu(x_E) \;-\; \partial^\mu\Lambda(x_E) \tag{S.55}$$

albeit subject to the periodicity condition

$$\partial^\mu\Lambda(\mathbf{x},x_4=0) \;=\; \partial^\mu\Lambda(\mathbf{x},x_4=\beta). \tag{S.56}$$

Consequently, the proper construction of the Euclidean Functional integral over the EM field configurations requires the same Fadde'ev–Popov gauge-fixing procedure as for $\mathcal{T}=0$ with suitable modifications to reflect the fields' periodicities. Thus, the EM Partition Function is

$$Z_{\rm EM} \;=\; C \iint\limits_{\rm periodic} \mathcal{D}[A^\mu(x_E)]\,\Delta_{\rm FP}e^{-S_E[A^\mu(x_E)]} \tag{S.57}$$

where the Euclidean action

$$S_E[A^\mu(x_E)] \;=\; \int\! d^3\mathbf{x}\int_0^\beta\! dx_4\,\Bigl\{\tfrac{1}{4}F_{\mu\nu}^2\;+\;\tfrac{1}{2\xi}(\partial A)^2\Bigr\} \tag{S.58}$$

includes the gauge-fixing term and the Fadde'ev–Popov determinant

$$\Delta_{\rm FP} \;=\; \mathrm{Det}(-\partial^2)_{\rm periodic} \tag{S.59}$$

takes into account the periodicity (S.56) of the finite-temperature gauge transformations. Finally, the normalization factor

$$C \;=\; \left[\iint\mathcal{D}[\omega(x_E)]\, e^{-\frac{1}{2\xi}\int\omega^2 d^4x_E}\right]^{-1} \tag{S.60}$$

compensating for the averaging over the gauge conditions $\partial_\mu A^\mu = \omega$ should also involve properly periodic $\omega(x_E)$.

For the free EM field, the Euclidean action functional (S.58) is quadratic and the functional integral (S.57) is purely Gaussian, but keeping in mind the Fadde'ev–Popov determinant factor $\Delta_{\rm FP}$, we have

$$Z_{\rm EM} \;=\; C\,\mathrm{Det}(-\partial^2)\times\left[\mathrm{Det}\left(-\partial^2\delta^{\mu\nu} + (1-\xi^{-1})\partial^\mu\partial^\nu\right)\right]^{-1/2} \tag{S.61}$$

where all the determinants are over periodic fields. Hence, in the momentum basis

$$\log Z_{\rm EM} \;=\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{k_4}\left\{\tfrac12\log(\xi^{-1}) \;+\; \log(k_E^2) \;-\; \tfrac12\log\det\left(k_E^2\delta^{\mu\nu} - (1-\xi^{-1})k_E^\mu k_E^\nu\right)\right\} \tag{S.62}$$

where the $4\times4$ matrix $(k_E^2\delta^{\mu\nu} - (1-\xi^{-1})k_E^\mu k_E^\nu)$ has three eigenvalues equal to $k_E^2$ (transverse eigenvectors) and one eigenvalue equal to $k_E^2/\xi$ (eigenvector parallel to the $k_E$). Thus,

$$\det\left(k_E^2\delta^{\mu\nu} - (1-\xi^{-1})k_E^\mu k_E^\nu\right) \;=\; \xi^{-1}(k_e^2)^4 \tag{S.63}$$

and therefore

$$\log Z_{\rm EM} \;=\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{k_4}\left\{-1\times\log(k_E^2)\right\}. \tag{S.64}$$

By comparison, a (real) scalar field has

$$\log Z_\phi \;=\int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{k_4}\left\{-\tfrac12\times\log(k_E^2+m^2)\right\}, \tag{S.65}$$

which means the EM field has the partition function of *two species* of a massless scalar —

or equivalently, two physical polarizations states for its own massless quanta, the photons. $\mathcal{Q}.\mathcal{E}.\mathcal{D}.$