1. As a warm-up exercise, consider an electrically charged heavy scalar field Φ interacting with a lighter ($m < M$) neutral scalar field φ according to

$$\mathcal{L} \;=\; \left[D_\mu\Phi^* D^\mu\Phi \;-\; M^2\Phi^*\Phi\right] \;+\; \left[\tfrac{1}{2}(\partial_\mu\varphi)^2 \;-\; \tfrac{1}{2}m^2\varphi^2\right] \;-\; \tfrac{1}{2}\lambda\,\Phi^*\Phi\,\varphi^2. \tag{1}$$

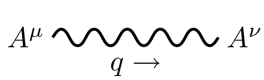
   Due to Coulomb forces, the charged particles Φ and antiparticles $\bar{\Phi}$ form hydrogen-like bound states; thanks to $\alpha \ll 1$, such bound states are essentially non-relativistic.

   Consider the ground $1S$ state of the $\Phi\bar{\Phi}$ bound system and calculate its lifetime against decay into two neutral scalars, $[\Phi\bar{\Phi}]_{1S} \to \varphi\varphi$. Ignore all the electromagnetic effects except for forming the bound state in the first place.

2. Consider the theory of photons (γ) interacting with electrons ($e^\mp$) and with charged scalar particles ($S^\mp$) according to

$$\mathcal{L} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \overline{\Psi}(i\not{D} - m)\Psi + \; D_\mu\Phi^* D^\mu\Phi \;-\; M^2\Phi^*\Phi. \tag{2}$$

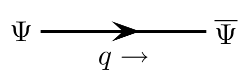
   The Feynman rules of this theory are as follows:

Photonic propagator: 

$$= \frac{-ig^{\mu\nu}}{q^2 + i0}\,, \tag{F.1}$$

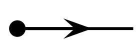
Incoming photon:  $= e_\mu(k,\lambda)$, (F.2)

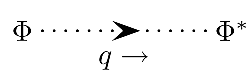
Outgoing photon:  $= e^*_\mu(k,\lambda)$, (F.3)

Electron propagator: 

$$= \frac{i}{\not{q} - m + i0}\,, \tag{F.4}$$

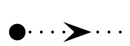
Incoming e^- or outgoing e^+:  $= u(p,s)$ or $v(p,s)$, (F.5)

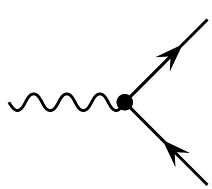
Outgoing e^- or incoming e^+:  $= \bar{u}(p,s)$ or $\bar{v}(p,s)$, (F.6)

Scalar propagator: 

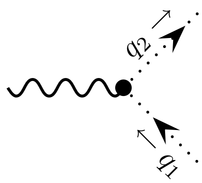
$$= \frac{i}{q^2 - M^2 + i0}\,, \tag{F.7}$$

Incoming S^- or outgoing S^+:  $= 1$, (F.8)

Outgoing S^- or incoming S^+:  $= 1$, (F.9)

QED vertex $ee\gamma$:  $= +ie\gamma^\mu$, (F.10)

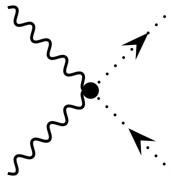
Scalar QED vertex $SS\gamma$:



$$= +ie(q_1 + q_2)^\mu \tag{F.11}$$

(mind the signs of q_1 and q_2),

Seagull vertex $SS\gamma\gamma$:



$$= +2ie^2\, g^{\mu\nu}. \tag{F.12}$$

(a) The QED Feynman rules (F.1–6) and (F.10) were derived in class. Derive the remaining rules (F.7–9) and (F.11–12).

(b) Given the above Feynman rules, calculate the tree-level matrix element for the scalar pair-production $e^-e^+ \to S^-S^+$. Hint: Mind the arrow directions on the scalar's dotted lines.

(c) Calculate the partial cross-section for the scalar pair production and compare its angular dependence with that of the $e^-e^+ \to \mu^-\mu^+$ process. Also, calculate the total cross-section $\sigma_{\text{tot}}(e^-e^+ \to S^-S^+)$ and compare its energy dependence to that of $\sigma_{\text{tot}}(e^-e^+ \to \mu^-\mu^+)$. For simplicity, neglect the electron's mass m.

3. Finally, consider a heavy *neutral* scalar particle S^0 with a Yukawa coupling to electrons — which in turn couple to the photons according to the usual QED rules, thus

$$\mathcal{L} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \overline{\Psi}(i\not{D} - m)\Psi + \;\left[\tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi \;-\; \tfrac{1}{2}M^2\varphi^2\right] \;+\; g\overline{\Psi}\varphi\Psi. \tag{3}$$

In this theory, an electron and a positron colliding with energy $E_{\text{c.m.}} > M$ may annihilate into one photon and one scalar particle, $e^-e^+ \to \gamma S^0$.

(a) Draw tree diagrams for the $e^-e^+ \to \gamma S^0$ process and write down the tree-level matrix element $\mathcal{M}(e^-e^+ \to \gamma S^0)$.

(b) Sum $|\mathcal{M}|^2$ over the photon's polarizations, average over the fermion's spins, and calculate the partial cross-section

$$\frac{d\sigma(e^-e^+ \to \gamma S^0)}{d\Omega_{\text{c.m.}}}\,.$$

For simplicity, neglect the electron's mass.