

Please do not waste time and paper by copying the posted homework solutions. If you need to use any homework result, simply reference the appropriate question or equation and go ahead. (For example, “According to homework 5.3(b), $M^{-1}(L)\gamma^\mu M(L) = L^\mu_\nu\gamma^\nu$ ”). Ditto for anything explicitly derived in class.

1. Consider the Yukawa theory of interacting scalar and Dirac fields. Two Dirac fermions Ψ^i form a doublet of the isospin symmetry group $SU(2)$ while three real scalar fields Φ^a form an isotriplet. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi^a\partial^\mu\Phi^a - \frac{1}{2}m^2\Phi^a\Phi^a + \bar{\Psi}^i(i\not{\partial} - M)\delta_{ij}\Psi^j - g\Phi^a\bar{\Psi}^i(\frac{1}{2}\tau^a)_{ij}\Psi^j - \frac{1}{24}\lambda(\Phi^a\Phi^a)^2 \quad (1)$$

where τ^a are the Pauli matrices for the isospin (denoted τ rather than σ to avoid confusion with the spin matrices).

- (a) Write down the classical equation of motion for all the fields.
- (b) Verify the isospin symmetry of the Lagrangian (1), write down the isospin’s Noether currents J_μ^a and verify that they are conserved.
- (c) Write down the isospin charges \hat{T}^a of the quantum field theory in terms of the bosonic and fermionic creation and annihilation operators and verify the isospin commutation relations: $[\hat{T}^a, \hat{T}^b] = i\epsilon^{abc}\hat{T}^c$.
2. Consider the Rarita–Schwinger spin-vector field $\Psi_a^\mu(x)$ which carries a 4–vector index μ as well as a Dirac spinor index a , although the latter is usually implicit in Dirac matrix notations. The free Rarita–Schwinger Lagrangian is rather complicated:

$$\mathcal{L} = \bar{\Psi}^\mu(i\not{\partial} - m)\Psi_\mu - \frac{i}{3}\bar{\Psi}^\mu(\gamma_\mu\not{\partial}_\nu + \partial_\mu\gamma_\nu)\Psi^\nu + \frac{1}{3}\bar{\Psi}^\mu\gamma_\mu(i\not{\partial} + m)\gamma_\nu\Psi^\nu, \quad (2)$$

but the free field equations are rather simple:

$$\gamma_\nu\Psi^\nu(x) = 0, \quad \partial_\nu\Psi^\nu(x) = 0, \quad (i\not{\partial} - m)\Psi^\nu(x) = 0. \quad (3)$$

(a) Derive the field equations (3) from the Lagrangian (2).

Hint: First, write the Euler–Lagrange equation for the $\Psi^\mu(x)$ field as $Z_{\mu\nu}\Psi^\nu = 0$ for some differential operator $Z_{\mu\nu}$, and then simplify and use $\gamma^\mu Z_{\mu\nu}\Psi^\nu = 0$ and $\partial^\mu Z_{\mu\nu}\Psi^\nu = 0$.

(b) Show that the particles created and annihilated by the quantum fields $\hat{\Psi}^\mu(x)$ and $\hat{\bar{\Psi}}^\mu(x)$ have spin $\frac{3}{2}$. Hint: Consider the plane-wave solutions of the field eqs. (3) and focus on the $\mathbf{p} = 0$ mode.

(c) Finally, describe how the Rarita–Schwinger field transforms under the orthochronous Lorentz transforms — including parity — and verify that the Lagrangian (2) transforms as a true scalar.