1. Consider a *massive* relativistic vector field $A^\mu(x)$ with the Lagrangian density

$$\mathcal{L} \;=\; -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \;+\; \tfrac{1}{2}m^2\,A_\mu A^\mu \;-\; A^\mu J_\mu \qquad (1)$$

(in $c=\hbar=1$ units) where $F_{\mu\nu} \stackrel{\rm def}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$ and the current $J^\mu(x)$ is a fixed source for the $A^\mu(x)$ field. Note that because of the mass term, the Lagrangian (1) is *not* gauge invariant.

(a) Derive the Euler–Lagrange field equations for the massive vector field $A^\mu(x)$.

(b) Show that this field equation *does not require* current conservation; however, if the current happens to satisfy $\partial_\mu J^\mu = 0$, then the field $A^\mu(x)$ satisfies

$$\partial_\mu A^\mu \;=\; 0 \qquad \text{and} \qquad (\partial^2 + m^2)A^\mu = J^\mu. \qquad (2)$$

Next, consider the Hamiltonian formalism for the massive vector field. Our first step in deriving this formalism is to identify the canonically conjugate "momentum" fields.

(c) Show that $\partial\mathcal{L}/\partial\dot{\mathbf{A}} = -\mathbf{E}$ but $\partial\mathcal{L}/\partial\dot{A}_0 \equiv 0$.

In other words, the canonically conjugate field to $\mathbf{A}(\mathbf{x})$ is $-\mathbf{E}(\mathbf{x})$ but the $A_0(\mathbf{x})$ does not have a canonical conjugate! Consequently,

$$H \;=\; -\int\! d^3\mathbf{x}\,\dot{\mathbf{A}}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}) \;-\; L. \qquad (3)$$

(d) Show that in terms of the $\mathbf{A}$, $\mathbf{E}$ and $A_0$ fields and their *space* derivatives,

$$H \;=\; \int\! d^3\mathbf{x}\left\{\tfrac{1}{2}\mathbf{E}^2 \;+\; A_0\,(J_0 - \nabla\cdot\mathbf{E}) \;-\; \tfrac{1}{2}m^2 A_0^2 \;+\; \tfrac{1}{2}\,(\nabla\times\mathbf{A})^2 \;+\; \tfrac{1}{2}m^2\mathbf{A}^2 \;-\; \mathbf{J}\cdot\mathbf{A}\right\}. \qquad (4)$$

Because the $A_0$ field does not have a canonical conjugate, the Hamiltonian formalism does not produce an equation for the time-dependence of this field. Instead, it gives us a time-independent equation relating the $A_0(\mathbf{x},t)$ to the values of other fields *at the same time* $t$.

Specifically, we have

$$\frac{\delta H}{\delta A_0(\mathbf{x})} \equiv \left.\frac{\partial \mathcal{H}}{\partial A_0}\right|_{\mathbf{x}} - \nabla\cdot \left.\frac{\partial \mathcal{H}}{\partial \nabla A_0}\right|_{\mathbf{x}} = 0. \tag{5}$$

At the same time, the vector fields **A** and **E** satisfy the Hamiltonian equations of motion,

$$\frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) = -\left.\frac{\delta H}{\delta \mathbf{E}(\mathbf{x})}\right|_t, \qquad \frac{\partial}{\partial t}\mathbf{E}(\mathbf{x},t) = +\left.\frac{\delta H}{\delta \mathbf{A}(\mathbf{x})}\right|_t. \tag{6}$$

(e) Write down the explicit form of all these equations.

(f) Finally, verify that the equations you have just written down are equivalent to the Euler–Lagrange equations you derived in question (a).

2. Later in this class, we shall learn how to construct the quantum electromagnetic fields $\hat{\mathbf{E}}(\mathbf{x},t)$ and $\hat{\mathbf{B}}(\mathbf{x},t)$ out of creation and annihilation operators in the photonic Fock space. For the moment, let us simply take it for granted that they obey the time-independent Maxwell eqs.

$$\nabla\cdot\hat{\mathbf{E}}(\mathbf{x},t) = \nabla\cdot\hat{\mathbf{B}}(\mathbf{x},t) = 0 \tag{7}$$

(we assume free EM fields, *i.e.* no electric charges or currents). In the Heisenberg picture, the quantum EM fields also obey the time-dependent Maxwell equations

$$\begin{aligned} \frac{\partial\hat{\mathbf{B}}}{\partial\mathbf{t}} &= -\nabla\times\hat{\mathbf{E}}, \\ \frac{\partial\hat{\mathbf{E}}}{\partial\mathbf{t}} &= +\nabla\times\hat{\mathbf{B}}, \end{aligned} \tag{8}$$

which follow from the free electromagnetic Hamiltonian

$$\hat{H}_{EM} = \int d^3\mathbf{x}\,\left(\tfrac{1}{2}\hat{\mathbf{E}}^2 + \tfrac{1}{2}\hat{\mathbf{B}}^2\right) \tag{9}$$

*and the equal-time commutation relations*

$$\begin{aligned} \left[\hat{E}_i(\mathbf{x},t), \hat{E}_j(\mathbf{x}',t'=t)\right] &= ???, \\ \left[\hat{B}_i(\mathbf{x},t), \hat{B}_j(\mathbf{x}',t'=t)\right] &= ???, \\ \left[\hat{E}_i(\mathbf{x},t), \hat{B}_j(\mathbf{x}',t'=t)\right] &= ???. \end{aligned}$$

Such commutation relations for the electromagnetic fields are completely determined by the consistency of eqs. (8) with the Hamiltonian (9), so **write them down**. Make sure your answer is consistent with the transversality of the fields, *i.e.*, with the time-independent Maxwell equations (7).

3. Finally, an exercise in using the bosonic commutation relations

$$[\hat{a}_\alpha, \hat{a}_\beta] \;=\; [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] \;=\; 0, \quad [\hat{a}_\alpha, \hat{a}^\dagger_\beta] \;=\; \delta_{\alpha,\beta}\,. \tag{10}$$

(a) Calculate the commutators $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma]$, $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}_\delta]$ and $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta]$.

I shall explain in class that the Hilbert space of the creation and annihilation operators $\hat{a}^\dagger_\alpha$ and $\hat{a}_\beta$ is the Fock space of (any number of) identical bosonic particles. In this space, operators of the type

$$\hat{A} \;=\; \sum_{\alpha,\beta} \langle\alpha|\,\hat{A}_1\,|\beta\rangle\; \hat{a}^\dagger_\alpha \hat{a}_\beta \tag{11}$$

describe net quantities which may be measured one particle at a time and then totaled up for all particles which happen to be present: On the subspace of $N$-particle states,

$$\hat{A}\Big|_N \;=\; \sum_{i=1}^{N} \hat{A}_1(i\frac{\text{the}}{\quad}\ \text{particle})\,. \tag{12}$$

where $\hat{A}_1$ is a one-particle operator (such as momentum or kinetic energy or angular momentum) and $\langle\alpha|\,\hat{A}_1\,|\beta\rangle$ in eq. (11) are its matrix elements in the one-particle Hilbert space. Later in class I shall explain the physical meaning of all kinds of Fock-space operators, but for the moment all you need is the rule (11) which constructs a Fock-space operator $\hat{A}$ for any one-particle operator $\hat{A}_1$.

(b) Consider three one-particle operators $\hat{A}_1$, $\hat{B}_1$ and $\hat{C}_1$ and the corresponding Fock-space operators

$$\hat{A} \;=\; \sum_{\alpha,\beta} \langle\alpha|\,\hat{A}_1\,|\beta\rangle\; \hat{a}^\dagger_\alpha \hat{a}_\beta\,, \quad \hat{B} \;=\; \sum_{\alpha,\beta} \langle\alpha|\,\hat{B}_1\,|\beta\rangle\; \hat{a}^\dagger_\alpha \hat{a}_\beta\,, \quad \hat{C} \;=\; \sum_{\alpha,\beta} \langle\alpha|\,\hat{C}_1\,|\beta\rangle\; \hat{a}^\dagger_\alpha \hat{a}_\beta\,. \tag{13}$$

Show that if $\hat{C}_1 = [\hat{A}_1, \hat{B}_1]$ then $\hat{C} = [\hat{A}, \hat{B}]$.