

1. The first exercise is about first-quantized *v.* second-quantized forms of one-body and two-body operators acting on identical bosons. In class, we wrote the wave function of an N -particle state $|\alpha_1, \dots, \alpha_N\rangle = |\{n_\beta\}\rangle$ as

$$\phi_{\alpha_1, \dots, \alpha_N}(\mathbf{x}_1, \dots, \mathbf{x}_N) = \sqrt{\frac{\prod_\beta n_\beta!}{N!}} \sum_{\substack{\text{distinct permutations} \\ (\tilde{\alpha}_1, \dots, \tilde{\alpha}_N) \text{ of } (\alpha_1, \dots, \alpha_N)}} \phi_{\tilde{\alpha}_1}(\mathbf{x}_1) \cdots \phi_{\tilde{\alpha}_N}(\mathbf{x}_N), \quad (1)$$

and we defined the annihilation operators \hat{a}_α according to

$$\hat{a}_\alpha |\{n_\beta\}\rangle = \sqrt{n_\alpha} |\{n'_\beta = n_\beta - \delta_{\alpha\beta}\}\rangle. \quad (2)$$

- (a) Consider an N -particle state $|N, \Psi\rangle$ with a completely generic totally-symmetric wave function $\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N)$. Show that the $(N-1)$ -particle state $|(N-1), \Psi'\rangle = \hat{a}_\gamma |N, \Psi\rangle$ has wave function

$$\Psi'(\mathbf{x}_1, \dots, \mathbf{x}_{N-1}) = \sqrt{N} \int d^3\mathbf{x}_N \phi_\gamma^*(\mathbf{x}_N) \Psi(\mathbf{x}_1, \dots, \mathbf{x}_{N-1}, \mathbf{x}_N). \quad (3)$$

Hint: First verify this formula for Ψ of the form (1), and then generalize to arbitrary (but totally-symmetric) Ψ by linearity.

Now consider a one-body operator \hat{R}_1 . In the first-quantized formalism \hat{R}_{tot} acts on N -particle states according to

$$\hat{R}_{\text{tot}}^{(1)} = \sum_{i=1}^N \hat{R}_1(i^{\text{th}} \text{ particle}) \quad (4)$$

while in the second-quantized formalism it becomes

$$\hat{R}_{\text{tot}}^{(2)} = \sum_{\alpha, \beta} \langle \alpha | \hat{R}_1 | \beta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta. \quad (5)$$

- (b) Use eq. (3) to verify that for any two N -particle states $\langle N, \Psi_1 |$ and $|N, \Psi_2\rangle$

$$\langle N, \Psi_1 | \hat{R}_{\text{tot}}^{(1)} |N, \Psi_2\rangle = \langle N, \Psi_1 | \hat{R}_{\text{tot}}^{(2)} |N, \Psi_2\rangle. \quad (6)$$

Hint: Use $\hat{R}_1 = \sum_{\alpha, \beta} |\alpha\rangle \langle \alpha | \hat{R}_1 | \beta \rangle \langle \beta |$.

Next, consider a two-body operator \hat{S}_2 which acts in the first-quantized formalism according to

$$\hat{S}_{\text{tot}}^{(1)} = \frac{1}{2} \sum_{i \neq j} \hat{S}_2(i \text{---th and } j \text{---th particles}) \quad (7)$$

and in the second-quantized formalism according to

$$\hat{S}_{\text{tot}}^{(2)} = \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha | \otimes \langle \beta | \hat{S}_2 | \gamma \rangle \otimes | \delta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta. \quad (8)$$

(c) Again, show that for any two N -particle states $\langle N, \Psi_1 |$ and $|N, \Psi_2\rangle$

$$\langle N, \Psi_1 | \hat{S}_{\text{tot}}^{(1)} |N, \Psi_2\rangle = \langle N, \Psi_1 | \hat{S}_{\text{tot}}^{(2)} |N, \Psi_2\rangle. \quad (9)$$

(d) Finally, let \hat{A}_1 be a one-body operator, let \hat{B}_2 and \hat{C}_2 be two-body operators, and let \hat{A} , \hat{B} and \hat{C} be the corresponding second-quantized operators defined similar to eqs. (5) and (8).

Show that if $\hat{C}_2 = \left[\left(\hat{A}_1(1^{\text{st}}) + \hat{A}_1(2^{\text{nd}}) \right), \hat{B}_2 \right]$ then $\hat{C} = [\hat{A}, \hat{B}]$

Hint: First, calculate the commutator $[\hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta, \hat{a}_\mu^\dagger \hat{a}_\nu]$.

2. The second problem is about coherent states of harmonic oscillators and free quantum fields. Let us start with a harmonic oscillator $\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a}$.

(a) For any complex number ξ we define a *coherent state* $|\xi\rangle \stackrel{\text{def}}{=} \exp(\xi \hat{a}^\dagger - \xi^* \hat{a}) |0\rangle$. Show that

$$|\xi\rangle = e^{-|\xi|^2/2} e^{\xi \hat{a}^\dagger} |0\rangle \quad \text{and} \quad \hat{a} |\xi\rangle = \xi |\xi\rangle. \quad (10)$$

(b) Calculate the uncertainties Δq and Δp for a coherent state $|\xi\rangle$ and verify their minimality: $\Delta q \Delta p = \frac{1}{2} \hbar$. Also, verify $\delta n = \sqrt{\bar{n}}$ where $\bar{n} \stackrel{\text{def}}{=} \langle \hat{n} \rangle = |\xi|^2$.

Hint: use $\hat{a} |\xi\rangle = \xi |\xi\rangle$ and $\langle \xi | \hat{a}^\dagger = \xi^* \langle \xi |$.

(c) Show that for $\xi(t) = \xi_0 e^{-e\omega t}$ the coherent state $|\xi(t)\rangle$ satisfies the time-dependent Schrödinger equation $i\hbar \frac{d}{dt} |\xi(t)\rangle = \hat{H} |\xi(t)\rangle$.

(d) The coherent states are not quite orthogonal to each other. Calculate their overlap $\langle \eta | \xi \rangle$.

Now consider coherent states of multi-oscillator systems and hence quantum fields. In particular, let us focus on the creation and annihilation fields $\hat{\Psi}^\dagger(\mathbf{x})$ and $\hat{\Psi}(\mathbf{x})$ for non-relativistic spinless bosons.

(e) Generalize (a) and construct coherent states $|\Phi\rangle$ which satisfy

$$\hat{\Psi}(\mathbf{x}) |\Phi\rangle = \Phi(\mathbf{x}) |\Phi\rangle \quad (11)$$

for any given classical complex field $\Phi(\mathbf{x})$.

(f) Show that for any such coherent state, $\Delta N = \sqrt{\bar{N}}$ where

$$\bar{N} \stackrel{\text{def}}{=} \langle \Phi | \hat{N} | \Phi \rangle = \int d\mathbf{x} |\Phi(\mathbf{x})|^2. \quad (12)$$

(g) Let

$$\hat{H} = \int d\mathbf{x} \left(\frac{\hbar^2}{2M} \nabla \hat{\Psi}^\dagger \cdot \nabla \hat{\Psi} + V(\mathbf{x}) \hat{\Psi}^\dagger \hat{\Psi} \right)$$

and show that for any classical field configuration $\Phi(\mathbf{x}, t)$ that satisfies the classical field equation

$$i\hbar \frac{\partial}{\partial t} \Phi(\mathbf{x}, t) = \left(-\frac{\hbar^2}{2M} \nabla^2 + V(\mathbf{x}) \right) \Phi(\mathbf{x}, t),$$

the time-dependent coherent state $|\Phi\rangle$ satisfies the true Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\Phi\rangle = \hat{H} |\Phi\rangle. \quad (13)$$

(h) Finally, show that the quantum overlap $|\langle \Phi_1 | \Phi_2 \rangle|^2$ between two different coherent states is exponentially small for any *macroscopic* difference $\delta\Phi(\mathbf{x}) = \Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})$ between the two field configurations.