

1. When an *exact* symmetry of a quantum field theory is spontaneously broken down, it gives rise to exactly massless Goldstone bosons. But when the spontaneously broken symmetry was only approximate to begin with, the would-be Goldstone bosons are no longer exactly massless but only relatively light. The best-known examples of such pseudo-Goldstone bosons are the pi-mesons π^\pm and π^0 , which are indeed much lighter than other hadrons. The Quantum Chromodynamics theory (QCD) of strong interactions has an approximate chiral isospin symmetry $SU(2)_L \times SU(2) \cong \text{Spin}(4)$ which would be exact if the two lightest quark flavors u and d were exactly massless; in reality, the current quark masses m_u and m_d do not exactly vanish but are small enough to be treated as a perturbation. Exact or approximate, the chiral isospin symmetry is spontaneously broken down to the ordinary isospin symmetry $SU(2) \cong \text{Spin}(3)$, and the 3 generators of the broken $\text{Spin}(4)/\text{Spin}(3)$ give rise to 3 (pseudo) Goldstone bosons π^\pm and π^0 .

QCD is a rather complicated theory, so it is often convenient to describe the physics of the spontaneously broken chiral symmetry in terms of a simpler effective theory such as the linear sigma model. This model has 4 real scalar fields; in terms of the unbroken isospin symmetry, we have an isosinglet $\sigma(x)$ and an isotriplet $\underline{\pi}(x)$ comprising $\pi^1(x)$, $\pi^2(x)$ and $\pi^3(x)$ (or equivalently, $\pi^0(x) \equiv \pi^3(x)$ and $\pi^\pm(x) \equiv (\pi^1(x) \pm i\pi^2(x))/\sqrt{2}$). The Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\underline{\pi})^2 - \frac{\lambda}{8}(\sigma^2 + \underline{\pi}^2 - f^2)^2 + \beta\sigma \quad (1)$$

is invariant under the $SO(4)$ rotations of the four fields, except for the last term which we take to be very small. (In QCD $\beta \sim \frac{m_u+m_d}{2f} \langle \bar{\Psi}\Psi \rangle$ which is indeed very small because the u and d quarks are very light.)

In class, we discussed this theory for $\beta = 0$ and showed that it has $SO(4)$ spontaneously broken to $SO(3)$ and hence 3 massless Goldstone bosons. In this exercise, we let $\beta > 0$ but $\beta \ll \lambda f^3$ to show how this leads to massive but light pions.

- (a) Show that the scalar potential of the linear sigma model with $\beta > 0$ has a unique

minimum at

$$\langle \underline{\pi} \rangle = 0 \quad \text{and} \quad \langle \sigma \rangle = f + \frac{\beta}{\lambda f^2} + O(\beta^2). \quad (2)$$

(b) Expand the fields around this minimum and show that the pions are light while the σ particle is much heavier. Specifically, $M_\pi^2 \approx (\beta/f)$ while $M_\sigma^2 \approx \lambda f^2$.

2. Our second exercise is about the Bogolyubov transform. Let $\hat{a}_{\mathbf{p}}$ and $\hat{a}_{\mathbf{p}}^\dagger$ be annihilation and creation operators satisfying the bosonic commutation relations, and let

$$\hat{b}_{\mathbf{p}} = \cosh(t_{\mathbf{p}})\hat{a}_{\mathbf{p}} + \sinh(t_{\mathbf{p}})\hat{a}_{-\mathbf{p}}^\dagger, \quad \hat{b}_{\mathbf{p}}^\dagger = \cosh(t_{\mathbf{p}})\hat{a}_{\mathbf{p}}^\dagger + \sinh(t_{\mathbf{p}})\hat{a}_{-\mathbf{p}} \quad (3)$$

for some arbitrary real parameters $t_{\mathbf{p}} = t_{-\mathbf{p}}$.

(a) Show that the $\hat{b}_{\mathbf{p}}$ and the $\hat{b}_{\mathbf{p}}^\dagger$ satisfy the same bosonic commutation relations as the $\hat{a}_{\mathbf{p}}$ and the $\hat{a}_{\mathbf{p}}^\dagger$.

(b) Consider a Hamiltonian of the form

$$\hat{H} = \sum_{\mathbf{p}} A_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{1}{2} \sum_{\mathbf{p}} B_{\mathbf{p}} \left(\hat{a}_{\mathbf{p}} \hat{a}_{-\mathbf{p}} + \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{-\mathbf{p}}^\dagger \right) \quad (4)$$

where $A_{\mathbf{p}} = A_{-\mathbf{p}}$ and $B_{\mathbf{p}} = B_{-\mathbf{p}}$. Show that as long as $|B_{\mathbf{p}}| < A_{\mathbf{p}} \forall \mathbf{p}$, this Hamiltonian can be “diagonalized” by means of a Bogolyubov transform (3). That is, for a suitable choice of the $t_{\mathbf{p}}$ parameters,

$$\hat{H} = \sum_{\mathbf{p}} \omega_{\mathbf{p}} \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}} + \text{const} \quad \text{where } \omega_{\mathbf{p}} = \sqrt{A_{\mathbf{p}}^2 - B_{\mathbf{p}}^2}. \quad (5)$$

In particular, for the liquid helium,

$$A_{\mathbf{p}} = \frac{\mathbf{p}^2}{2M} + n_0 \lambda(\mathbf{p}) \quad \text{and} \quad B_{\mathbf{p}} = n_0 \lambda(\mathbf{p}) \quad \implies \quad \omega_{\mathbf{p}} = \sqrt{\frac{\mathbf{p}^2}{2M} \left(\frac{\mathbf{p}^2}{2M} + n_0 \lambda(\mathbf{p}) \right)}. \quad (6)$$

3. The rest of this homework is about a charged relativistic scalar field $\Phi(x)$. A conserved charge implies a complex field with a $U(1)$ symmetry $\Phi(x) \mapsto e^{i\theta} \Phi(x)$ which gives rise to

a conserved Noether current

$$J^\mu = i\Phi^* \partial^\mu \Phi - i(\partial^\mu \Phi^*) \Phi. \quad (7)$$

For simplicity, let the Φ field be free, thus

$$\mathcal{L} = \partial^\mu \Phi^* \partial_\mu \Phi - m^2 \Phi^* \Phi. \quad (8)$$

In the Hamiltonian formalism, we trade the time derivatives $\partial_0 \Phi(x)$ and $\partial_0 \Phi^*(x)$ for the canonically conjugate fields $\Pi^*(x)$ and $\Pi(x)$. (Note that for complex fields $\Pi(\mathbf{x})$ is canonically conjugate to the $\Phi^*(\mathbf{x})$ while $\Pi^*(\mathbf{x})$ is canonically conjugate to the $\Phi(\mathbf{x})$.) Canonical quantization of this system yields non-hermitian quantum fields $\hat{\Phi}(x) \neq \hat{\Phi}^\dagger(x)$ and $\hat{\Pi}(x) \neq \hat{\Pi}^\dagger(x)$ and the Hamiltonian operator

$$\hat{H} = \int d^3\mathbf{x} \left(\hat{\Pi}^\dagger \hat{\Pi} + \nabla \hat{\Phi}^\dagger \cdot \nabla \hat{\Phi} + m^2 \hat{\Phi}^\dagger \hat{\Phi} \right). \quad (9)$$

- (a) Derive the Hamiltonian (9) and write down the equal-time commutation relations between the quantum fields $\hat{\Phi}(\mathbf{x})$, $\hat{\Phi}^\dagger(\mathbf{x})$, $\hat{\Pi}(\mathbf{x})$ and $\hat{\Pi}^\dagger(\mathbf{x})$.

The plane-wave modes

$$\hat{\Phi}_{\mathbf{p}} = \int d^3\mathbf{x} e^{-i\mathbf{p}\mathbf{x}} \hat{\Phi}(\mathbf{x}), \quad \hat{\Pi}_{\mathbf{p}} = \int d^3\mathbf{x} e^{-i\mathbf{p}\mathbf{x}} \hat{\Pi}(\mathbf{x}), \quad (10)$$

of non-hermitian fields are completely independent of each other, thus $\hat{\Phi}_{\mathbf{p}}^\dagger \neq \hat{\Phi}_{-\mathbf{p}}$ and $\hat{\Pi}_{\mathbf{p}}^\dagger \neq \hat{\Pi}_{-\mathbf{p}}$. Consequently, we have two independent species of creation and annihilation operators; in the relativistic normalization

$$\begin{aligned} \hat{a}_{\mathbf{p}} &\stackrel{\text{def}}{=} E_{\mathbf{p}} \hat{\Phi}_{\mathbf{p}} + i\hat{\Pi}_{\mathbf{p}}, & \hat{a}_{\mathbf{p}}^\dagger &\stackrel{\text{def}}{=} E_{\mathbf{p}} \hat{\Phi}_{\mathbf{p}}^\dagger - i\hat{\Pi}_{\mathbf{p}}^\dagger, \\ \hat{b}_{\mathbf{p}} &\stackrel{\text{def}}{=} E_{\mathbf{p}} \hat{\Phi}_{-\mathbf{p}}^\dagger + i\hat{\Pi}_{-\mathbf{p}}^\dagger, & \hat{b}_{\mathbf{p}}^\dagger &\stackrel{\text{def}}{=} E_{\mathbf{p}} \hat{\Phi}_{-\mathbf{p}} - i\hat{\Pi}_{-\mathbf{p}}, \end{aligned} \quad (11)$$

where $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$.

- (b) Verify the bosonic commutation relations (at equal times) between the annihilation operators $\hat{a}_{\mathbf{p}}$ and $\hat{b}_{\mathbf{p}}$ and the corresponding creation operators $\hat{a}_{\mathbf{p}}^\dagger$ and $\hat{b}_{\mathbf{p}}^\dagger$.
- (c) Show that the Hamiltonian of the free charged fields is

$$\hat{H} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \left(E_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + E_{\mathbf{p}} \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}} \right) + \text{const.} \quad (12)$$

Next, consider the charge operator $\hat{Q} = \int d^3\mathbf{x} \hat{J}_0(\mathbf{x})$.

- (d) Show that for the system at hand

$$\hat{Q} = \int d^3\mathbf{x} \left(\frac{i}{2} \{ \hat{\Pi}^\dagger, \hat{\Phi} \} - \frac{i}{2} \{ \hat{\Pi}, \hat{\Phi}^\dagger \} \right) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \left(\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} - \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}} \right). \quad (13)$$

Actually, the classical formula (7) for the current $J_\mu(x)$ determines eq. (13) only up to ordering of the non-commuting operators $\hat{\Pi}(\mathbf{x})$ and $\hat{\Phi}^\dagger(\mathbf{x})$ (and likewise of the $\hat{\Pi}^\dagger(\mathbf{x})$ and $\hat{\Phi}(\mathbf{x})$). The anti-commutators in eq. (13) provide a solution to this ordering ambiguity, but any other ordering would be just as legitimate. The net effect of changing operator ordering in \hat{J}_0 amounts to changing the total charge \hat{Q} by an infinite constant (prove this!). The specific ordering in eq. (13) provides for the neutrality of the vacuum state.

Now consider the stress-energy tensor of the charged field. Classically, Noether theorem gives

$$T^{\mu\nu} = \partial^\mu \Phi^* \partial^\nu \Phi + \partial^\mu \Phi \partial^\nu \Phi^* - g^{\mu\nu} \mathcal{L}. \quad (14)$$

Quantization of this formula is straightforward (modulo ordering ambiguity); for example, $\hat{\mathcal{H}} \equiv \hat{T}^{00}$ is precisely the integrand on the right hand side of eq. (9).

- (e) Show that the total mechanical momentum operator of the fields is

$$\hat{\mathbf{P}}_{\text{mech}} \stackrel{\text{def}}{=} \int d^3\mathbf{x} \hat{T}^{0,i} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \mathbf{p} \left(\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}} \right) \quad (15)$$

Physically, eqs. (15), (12) and (13) show that a complex field $\Phi(x)$ describes both a particle and its antiparticle; they have exactly the same rest mass m but exactly opposite charges ± 1 .

4. Finally, consider the time-dependence of the free charged fields $\hat{\Phi}(\mathbf{x}, t)$ and $\hat{\Phi}^\dagger(\mathbf{x}, t)$.

- (a) Compare the Schrödinger and the Heisenberg pictures of the creation and annihilation operators (11) and show that for the free Hamiltonian (12),

$$\hat{a}_{\mathbf{p}}^H(t) = e^{-itE_{\mathbf{p}}}\hat{a}_{\mathbf{p}}^S, \quad \hat{b}_{\mathbf{p}}^H(t) = e^{-itE_{\mathbf{p}}}\hat{b}_{\mathbf{p}}^S, \quad \hat{a}_{\mathbf{p}}^{\dagger H}(t) = e^{+itE_{\mathbf{p}}}\hat{a}_{\mathbf{p}}^{\dagger S}, \quad \hat{b}_{\mathbf{p}}^{\dagger H}(t) = e^{+itE_{\mathbf{p}}}\hat{b}_{\mathbf{p}}^{\dagger S}. \quad (16)$$

- (b) Now assemble the Heisenberg-picture quantum fields in terms of the operators (16) and show that in relativistic notations

$$\begin{aligned} \hat{\Phi}^H(x) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \left(e^{-ipx} \hat{a}_{\mathbf{p}}^S + e^{+ipx} \hat{b}_{\mathbf{p}}^{\dagger S} \right)_{p^0=E_{\mathbf{p}}}, \\ \hat{\Phi}^{\dagger H}(x) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \left(e^{-ipx} \hat{b}_{\mathbf{p}}^S + e^{+ipx} \hat{a}_{\mathbf{p}}^{\dagger S} \right)_{p^0=E_{\mathbf{p}}}, \end{aligned} \quad (17)$$

where $px \stackrel{\text{def}}{=} p^\mu x_\mu = E_{\mathbf{p}}t - \mathbf{p}\mathbf{x}$.

Eqs. (17) allow us to derive the commutation relations between the quantum fields at un-equal times.

- (c) Show that $[\hat{\Phi}^H(x), \hat{\Phi}^H(x')] = 0$ and $[\hat{\Phi}^{\dagger H}(x), \hat{\Phi}^{\dagger H}(x')] = 0$ even for un-equal times $x_0 \neq x'_0$.

- (d) Show that for un-equal times $x_0 \neq x'_0$,

$$[\hat{\Phi}^H(x), \hat{\Phi}^{\dagger H}(x')] = D(x - x') - D(x' - x) \quad (18)$$

where

$$D(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} e^{-ipx} \Big|_{p^0=E_{\mathbf{p}}}. \quad (19)$$

In class, we shall see that $D(x - x')$ is invariant under orthochronous Lorentz transformations. Consequently, for space-like $(x - x')$, $D(x - x') = D(x' - x)$ and therefore $[\hat{\Phi}^H(x), \hat{\Phi}^{\dagger H}(x')] = 0$.

In general, for any consistent quantum field theory, for any measurable local operators $\hat{O}_1(x_1)$ and $\hat{O}_2(x_2)$ constructed from the quantum fields and their space or time derivatives at respective points x_1 and x_2 , the commutator $[\hat{O}_1(x_1), \hat{O}_2(x_2)]$ must vanish for any space-like interval $(x_1 - x_2)^2 < 0$.