1. First, a few exercises concerning the plane-wave solutions $e^{-ipx}u(p,s)$ and $e^{+ipx}v(p,x)$ of the Dirac equation.

   (a) Show that

$$\sum_{s=1,2} u_a(p,s)\bar{u}_b(p,s) \;=\; (\not{p}+m)_{ab} \quad\text{and}\quad \sum_{s=1,2} v_a(p,s)\bar{v}_b(p,s) \;=\; (\not{p}-m)_{ab}\,. \tag{1}$$

   (b) Prove the Gordon identity

$$\bar{u}(p',s')\gamma^\mu u(p.s) \;=\; \frac{(p'+p)^\mu}{2m}\,\bar{u}(p's')u(p,s) \;+\; \frac{i(p'-p)_\nu}{m}\,\bar{u}(p's')S^{\mu\nu}u(p,s). \tag{2}$$

   Hint: First, use Dirac equations for the u and the $\bar{u}'$ to show that $2m\bar{u}'\gamma^\mu u = \bar{u}'(\not{p}'\gamma^\mu + \gamma^\mu\not{p})u$.

   (c) Generalize the Gordon identity to $\bar{u}'\gamma^\mu v$, $\bar{v}'\gamma^\mu u$ and $\bar{v}'\gamma^\mu v$.

2. The second problem concerns finite representations of the Lorentz symmetry, or rather $\mathrm{Spin}(3,1) \cong SL(2,\mathbf{C})$. Consider the Lorentz generators $\hat{J}^{\mu\nu}$: In 3–index notations, the $\hat{J}^{ij} = \epsilon^{ij\ell}\hat{J}^\ell$ generate ordinary rotations while the $\hat{J}^{0i} = -\hat{J}^{i0} = \hat{K}^i$ generate the Lorentz boosts. Let

$$\hat{\mathbf{J}}_\pm \;=\; \tfrac{1}{2}\big(\hat{\mathbf{J}} \;\pm\; i\hat{\mathbf{K}}\big). \tag{3}$$

   (a) Show that the $\hat{\mathbf{J}}_+$ and the $\hat{\mathbf{J}}_-$ commute with each other and that each satisfies the commutations relations of an angular momentum, $[\hat{J}^k_\pm, \hat{J}^\ell_\pm] = i\epsilon^{k\ell m}\hat{J}^m_\pm$.

   The "angular momentum" $\hat{\mathbf{J}}_+$ is non-hermitian and hence its finite irreducible representations are non-unitary analytic continuations of the spin–j representations of a hermitian $\hat{\mathbf{J}}$. The same is true for the $\hat{\mathbf{J}}_- = \hat{\mathbf{J}}_+^\dagger$. Thus altogether, the finite irreducible representations of the Lorentz algebra are specified by two integer or half-integer 'spins' j_+ and j_-.

The simplest non-trivial representations of the Lorentz algebra are $(j_+ = \frac{1}{2}, j_- = 0)$ — the left-handed Weyl spinor where $\hat{\mathbf{J}}$ acts as $\frac{1}{2}\boldsymbol{\sigma}$ and $\hat{\mathbf{K}}$ as $-\frac{i}{2}\boldsymbol{\sigma}$, and $(j_+ = 0, j_- = \frac{1}{2})$ — the right-handed Weyl spinor where $\hat{\mathbf{J}}$ also acts as $\frac{1}{2}\boldsymbol{\sigma}$ but $\hat{\mathbf{K}}$ acts as $+\frac{i}{2}\boldsymbol{\sigma}$. Together, the two Weyl spinors comprise the Dirac spinor. From the $SL(2,\mathbf{C})$ point if view, the left-handed Weyl spinor is the doublet representation $\mathbf{2}$ which defines the $SL(2,\mathbf{C})$ group while the right-handed Weyl spinor is the conjugate doublet $\bar{\mathbf{2}}$. As discussed in class, the Weyl spinors transform according to

$$\psi^L_\alpha \;\mapsto\; M_\alpha^{\ \beta}\psi^L_\beta \qquad \text{and} \quad (\sigma_2\psi^R)_{\dot\alpha} \;\mapsto\; M^{*\dot\beta}_{\dot\alpha}(\sigma_2\psi^R)_{\dot\beta} \tag{4}$$

where $M \equiv M_L$ and $\sigma_2 M^* \sigma_2 = M_R$.

A generic (j_+, j_-) representation of the Lorentz algebra becomes in the $SL(2,\mathbf{C})$ terms a tensor $\Phi_{\alpha_1\ldots\alpha_{(2j_+)},\dot\gamma_1\ldots\dot\gamma_{(2j_-)}}$, totally symmetric in its $2j_+$ un-dotted indices $\alpha_1,\ldots,\alpha_{(2j_+)}$ and separately totally symmetric in its $2j_-$ dotted indices $\dot\gamma_1,\ldots,\dot\gamma_{(2j_-)}$; it transforms according to

$$\Phi_{\alpha_1\ldots\alpha_{(2j_+)},\dot\gamma_1\ldots\dot\gamma_{(2j_-)}} \;\mapsto\; M_{\alpha_1}^{\ \beta_1}\cdots M_{\alpha_{(2j_+)}}^{\ \beta_{(2j_+)}}\; M^{*\,\dot\delta_1}_{\dot\gamma_1}\cdots U^{*\,\dot\delta_{(2j_-)}}_{\dot\gamma_{(2j_-)}}\;\Phi_{\beta_1\ldots\beta_{(2j_+)},\dot\delta_1\ldots\dot\delta_{(2j_-)}}\,. \tag{5}$$

The vector representation of the Lorentz group has $j_+ = j_- = \frac{1}{2}$. To cast the action of the Lorentz group in $SL(2,\mathbf{C})$ terms (5), we define

$$X_{\alpha\dot\gamma} \;=\; X_\mu\sigma^\mu_{\alpha\dot\gamma} \;=\; X_0\delta_{\alpha\dot\gamma} \;-\; \mathbf{X}\cdot\boldsymbol{\sigma}_{\alpha\dot\gamma}\,, \tag{6}$$

or in 2×2 matrix notations, $X_\mu\sigma^\mu = X_0 - \mathbf{X}\cdot\boldsymbol{\sigma}$ where σ^0 is the unit matrix while σ^1, σ^2 and σ^3 are the Pauli matrices. In the $SL(2,\mathbf{C})$ terms, we have

$$X'_{\alpha\dot\gamma} \;=\; M_\alpha^{\ \beta} M^{*\dot\delta}_{\dot\gamma} X_{\gamma\dot\delta} \qquad i.e., \quad X'_\mu\sigma^\mu \;=\; M(X_\mu\sigma^\mu)M^\dagger. \tag{7}$$

(b) Show that for any $SL(2,\mathbf{C})$ matrix M, eq. (7) defines an orthochronous Lorentz transform $X'_\mu = L_\mu^{\ \nu}(M)X_\nu$. (Hint: prove and use $\det(X_\mu\sigma^\mu) = X^2 \equiv X_\mu X^\mu$).

   ∗ For extra challenge, show that L is proper, *i.e.* $\det(L) = +1$.

(c) Verify the group law, $L(M_2M_1) = L(M_2)L(M_1)$.

(d) Verify explicitly that for $M = \exp\bigl(-\frac{i}{2}\theta\,\mathbf{n}\cdot\boldsymbol{\sigma}\bigr)$, $L(M)$ is a rotation by angle θ around axis $\mathbf{n}$ while for $M = \exp\bigl(-\frac{1}{2}r\,\mathbf{n}\cdot\boldsymbol{\sigma}\bigr)$, $L(M)$ is a boost of rapidity r ($\beta = \tanh r$, $\gamma = \cosh r$) in the direction $\mathbf{n}$.

In general, any (j_+, j_-) multiplet of the $SL(2,\mathbf{C})$ with integer net spin $j_+ + j_-$ is equivalent to some kind of a Lorentz tensor. (Here, we include the scalar and the vector among the tensors.) For example, the $(1,1)$ multiplet is equivalent to a symmetric, traceless 2–index tensor $T^{\mu\nu} = T^{\nu\mu}$, $T^\mu_\mu = 0$. For $j_+ \neq j_-$ the representation is complex, but one can make a real tensor by combining two multiplets with opposite j_+ and j_-, for example the $(1,0)$ and $(0,1)$ multiplets are together equivalent to an antisymmetric 2–index tensor $F^{\mu\nu} = -F^{\nu\mu}$.

(e) Verify the above examples.
Hint: For any angular momentum $(j = \frac{1}{2}) \otimes (j = \frac{1}{2}) = (j = 1) \oplus (j = 0)$.

The $SL(2,\mathbf{C})$ multiplets with half-integer $j_+ + j_-$ are equivalent to Lorentz spinors or spin-tensors which carry one Weyl index as well as 0, 1 or more 4–vector indices and transform according to

$$\psi_\alpha^{\mu,\ldots,\nu} \;\mapsto\; M_\alpha^{\ \beta}(L) L^\mu_{\ \kappa} \cdots L^\nu_{\ \lambda}\, \psi_\beta^{\kappa,\ldots,\lambda} \qquad \text{or} \qquad \psi_{\dot\alpha}^{\mu,\ldots,\nu} \;\mapsto\; M_{\dot\alpha}^{*\dot\beta}(L) L^\mu_{\ \kappa} \cdots L^\nu_{\ \lambda}\, \psi_{\dot\beta}^{\kappa,\ldots,\lambda}. \tag{8}$$

(f) Show that the $(1,\frac{1}{2})$ and $(\frac{1}{2},1)$ multiplets are together equivalent to the Rarita–Schwinger spin-vector Ψ^μ_a which has one Dirac index a and one 4–vector index μ and satisfies a Lorentz-covariant constraint $\gamma_\mu\Psi^\mu = 0$.

3. Finally, consider the relation between Lorentz transformations of the fields and of the particles. In mechanics (classical or quantum), one must distinguish between two opposite kinds of rotations, namely coordinate-frame rotations of bodies and body-frame rotations of coordinate systems. For the Lorentz transformations of fields and particles, there is a similar distinction between the particle-frame and field-frame Lorentz transforms.

For example, consider a real (hermitian) scalar quantum field

$$\hat\Phi(x) \;=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_\mathbf{p}} \left[e^{-ipx}\,\hat a(p) \;+\; e^{+ipx}\,\hat a^\dagger(p) \right]_{p^0\equiv E_\mathbf{p}} \tag{9}$$

(where $\hat a(p)$ stands for the $\hat a_\mathbf{p}(t=0)$ and ditto for the $\hat a^\dagger(p)$). A field-frame Lorentz transform L acts on this field according to

$$\hat{\Phi}'(x') \;\equiv\; \hat{\mathcal{D}}^\dagger(L)\,\hat{\Phi}(x')\,\hat{\mathcal{D}}(L) \;=\; \hat{\Phi}(x=L^{-1}x') \tag{10}$$

while the corresponding particle-frame transform acts precisely in reverse:

$$\hat{\mathcal{D}}(L)\,\hat{\Phi}(x)\,\hat{\mathcal{D}}^\dagger(L) \;=\; \hat{\Phi}(Lx). \tag{11}$$

In both cases $\hat{\mathcal{D}}(L) = \exp\bigl(\frac{i}{2}\theta_{\alpha\beta}\hat{J}^{\alpha\beta}\bigr)$ is a unitary operator representing the Lorentz transform L in the Fock space of the quantum field theory.

(a) Show that (11) implies

$$\hat{\mathcal{D}}(L)\,\hat{a}(p)\,\hat{\mathcal{D}}^\dagger(L) \;=\; \hat{a}(Lp), \qquad \hat{\mathcal{D}}(L)\,\hat{a}^\dagger(p)\,\hat{\mathcal{D}}^\dagger(L) \;=\; \hat{a}^\dagger(Lp), \tag{12}$$

and therefore

$$\hat{\mathcal{D}}(L)\,|p\rangle \;=\; |Lp\rangle\,, \quad \hat{\mathcal{D}}(L)\,|p_1,p_2\rangle \;=\; |Lp_1,Lp_2\rangle\,, \quad \textit{etc., etc.} \tag{13}$$

thus *particle*-frame Lorentz transform.

Now consider a generic Lorentz multiplet of quantum fields $\hat{\phi}_A(x)$ which transform into each other according to

$$\hat{\phi}'_A(x') \;\equiv\; \hat{\mathcal{D}}^\dagger(L)\,\hat{\phi}_A(x')\,\hat{\mathcal{D}}(L) \;=\sum_B M_A^{\ B}(L)\,\hat{\phi}_B(x=L^{-1}x') \tag{14}$$

in the field frame, or

$$\hat{\mathcal{D}}(L)\,\hat{\phi}_A(x)\,\hat{\mathcal{D}}^\dagger(L) \;=\sum_B M_A^{\ B}(L^{-1})\,\hat{\phi}_B(Lx) \tag{15}$$

in the particle frame. In both frames, the matrices $M_A^{\ B}(L)$ form a finite but non-unitary representation of the Lorentz group while the Fock-space operators $\mathcal{D}(L)$ form a unitary but infinite representation.

(b) Verify that formula (15) is consistent with the same group law for both the field-multiplet and the Fock–space representations, $M_A^{\ C}(L_1L_2) = \sum_B M_A^{\ B}(L_1)M_B^{\ C}(L_2)$ while $\hat{\mathcal{D}}(L_2L_1) = \hat{\mathcal{D}}(L_2)\,\hat{\mathcal{D}}(L_1)$.

A free (complex) quantum field comprises particle and antiparticle creation and annihilation operators according to

$$\begin{aligned}
\hat{\phi}_A(x) &= \int\frac{d^3\mathbf{p}}{(2\pi)^3\,2E_{\mathbf{p}}}\sum_s \left[e^{-ipx} f_A(p,s)\,\hat{a}(p,s) \;+\; e^{+ipx} h_A(p,s)\,\hat{b}^\dagger(p,s)\right]_{p^0\equiv E_{\mathbf{p}}} \\
\hat{\phi}^\dagger_{\bar{A}}(x) &= \int\frac{d^3\mathbf{p}}{(2\pi)^3\,2E_{\mathbf{p}}}\sum_s \left[e^{-ipx} h^*_{\bar{A}}(p,s)\,\hat{b}(p,s) \;+\; e^{+ipx} f^*_{\bar{A}}(p,s)\,\hat{a}^\dagger(p,s)\right]_{p^0\equiv E_{\mathbf{p}}}
\end{aligned} \tag{16}$$

where $e^{-ipx}f_A(p,s)$ and $e^{+ipx}h_A(p,s)$ are independent plane-wave solutions of the free field equation for the ϕ_A, whatever that might be. For the real (*i.e.*, non-hermitian) fields, there are similar formulae where $h_A(p,s) = f^*_{\bar{A}}(p,s)$, $\hat{b}(p,s) = \hat{a}(p,s)$ and $\hat{b}^\dagger(p,s) = \hat{a}^\dagger(p,s)$, *i.e.*, the particles are their own antiparticles.

(c) A particle-frame Lorentz transform should act on particle or antiparticle quantum numbers according to

$$\hat{\mathcal{D}}(L)\;|p,\pm,s\rangle \;=\sum_{s'} C_{s,s'}(L,p)\;\big|Lp,\pm,s'\big\rangle\,. \tag{17}$$

Show that eqs. (15) and (17) are consistent with each other if and only if

$$\begin{aligned}
f_A(Lp,s') &= \sum_B\sum_s M_A^{\ B}(L)\,C^*_{s,s'}(L,p)\,f_B(p,s), \\
h_A(Lp,s') &= \sum_b\sum_s M_A^{\ B}(L)\,C_{s,s'}(L,p)\,h_B(p,s).
\end{aligned} \tag{18}$$