

1. Consider the matrix  $\gamma^5 \stackrel{\text{def}}{=} i\gamma^0\gamma^1\gamma^2\gamma^3$ .

(a) Show that  $\gamma^5$  anticommutes with each of the  $\gamma^\mu$  matrices,  $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ .

(b) Show that  $\gamma^5$  is hermitian and that  $(\gamma^5)^2 = 1$ .

(c) Show that  $\gamma^5 = (-i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$  and  $\gamma^{[\kappa\gamma^\lambda\gamma^\mu\gamma^\nu]} = -i\epsilon^{\kappa\lambda\mu\nu}\gamma^5$ .  
(Sign convention:  $\epsilon^{0123} = +1$ ,  $\epsilon_{0123} = -1$ .)

(d) Show that  $\gamma^{[\lambda\gamma^\mu\gamma^\nu]} = i\epsilon^{\kappa\lambda\mu\nu}\gamma_\kappa\gamma^5$ .

(e) Show that any  $4 \times 4$  matrix  $\Gamma$  is a unique linear combination of the following 16 matrices:  $1$ ,  $\gamma^\mu$ ,  $\gamma^{[\mu\gamma^\nu]}$ ,  $\gamma^5\gamma^\mu$  and  $\gamma^5$ .

Under continuous Lorentz symmetries, a Dirac spinor field  $\Psi(x)$  transforms according to  $\Psi'(x') = M(L)\Psi(x = L^{-1}x')$  where  $M(L = e^\theta) = \exp(-\frac{i}{2}\theta_{\alpha\beta}S^{\alpha\beta})$ . Consider the transformation rules for the independent bilinear products  $\bar{\Psi}\Gamma\Psi$  of a Dirac field and its conjugate  $\bar{\Psi}(x)$ , namely (*cf.* (e))

$$S = \bar{\Psi}\Psi, \quad V^\mu = \bar{\Psi}\gamma^\mu\Psi, \quad T^{\mu\nu} = \bar{\Psi}\gamma^{[\mu}\gamma^{\nu]}\Psi, \quad A^\mu = \bar{\Psi}\gamma^5\gamma^\mu\Psi \quad \text{and} \quad P = \bar{\Psi}\gamma^5\Psi. \quad (1)$$

(f) Show that under *continuous* Lorentz symmetries, the  $S$  and the  $P$  transform as scalars, the  $V^\mu$  and the  $A^\mu$  as vectors and the  $T^{\mu\nu}$  as an antisymmetric tensor.

2. Under the *parity* symmetry  $\mathcal{P} : (\mathbf{x}, t) \mapsto (-\mathbf{x}, t)$ , Dirac spinor fields transform according to

$$\hat{\mathcal{P}}\hat{\Psi}(\mathbf{x}, t)\hat{\mathcal{P}} \equiv \hat{\Psi}'(\mathbf{x}, t) = \pm\gamma^0\hat{\Psi}(-\mathbf{x}, t) \quad (2)$$

where the overall sign depends on the so-called *intrinsic parity* of a particular Dirac field. Note:  $\hat{\mathcal{P}}$  here is a unitary operator in the fermionic Fock space; by nature of the parity symmetry,  $\hat{\mathcal{P}}^2 = 1$ .

(a) Verify the covariance of the Dirac equation under this symmetry.

(b) Find the transformation rules of the bilinears (1) under parity and show that while  $S$  is a true scalar and  $V$  is a true (polar) vector,  $P$  is a pseudoscalar and  $A$  is an axial vector.

3. Next, consider the charge-conjugation properties of Dirac bilinears  $\bar{\Psi}\Gamma\Psi$ . To avoid operator ordering problems, take  $\Psi(x)$  and  $\Psi^\dagger(x)$  to be “classical” fermionic fields which *anticommute* with each other,  $\Psi_\alpha\Psi^\dagger\beta = -\Psi^\dagger\beta\Psi_\alpha$ .

(a) Show that  $\hat{\mathcal{C}}\hat{\Psi}\Gamma\hat{\Psi}\hat{\mathcal{C}} = \hat{\Psi}\Gamma^c\hat{\Psi}$  where  $\Gamma^c = \gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2$ .

(b) Calculate  $\Gamma^c$  for all 16 independent matrices  $\Gamma$  and find out which Dirac bilinears are  $\mathcal{C}$ -even and which are  $\mathcal{C}$ -odd.

4. In theories involving both bosons and fermions, one often has to combine commutation and anti-commutation relations of various operators, depending on the overall statistics of the operators involved. For that purpose, it is useful to define a ‘mixed’ commutator bracket

$$[\hat{A}, \hat{B}] \stackrel{\text{def}}{=} \hat{A}\hat{B} - (-1)^{AB}\hat{B}\hat{A} \quad (3)$$

where  $(-1)^{AB}$  is  $-1$  if both  $\hat{A}$  and  $\hat{B}$  have overall Fermi statistics (*i.e.*, each comprises an odd number of *fermionic* creation/annihilation operators — the number of bosonic creation/annihilation operators does not matter) and  $+1$  in all other cases.

(a) Verify the Leibniz rule for the mixed brackets:  $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + (-1)^{AB}\hat{B}[\hat{A}, \hat{C}]$  and write down a similar rule for the  $[\hat{A}\hat{B}, \hat{C}]$ .

(b) Similarly, express  $[\hat{A}\hat{B}, \hat{C}\hat{D}]$  in terms of appropriate mixed brackets of  $\hat{A}$  or  $\hat{B}$  with  $\hat{C}$  or  $\hat{D}$ .

(c) Prove the ‘mixed’ Jacobi identity

$$(-1)^{CA}[\hat{A}, [\hat{B}, \hat{C}]] + (-1)^{AB}[\hat{B}, [\hat{C}, \hat{A}]] + (-1)^{BC}[\hat{C}, [\hat{A}, \hat{B}]] = 0. \quad (4)$$

In other words (and notations),

$$\begin{aligned}
[\hat{B}_1, [\hat{B}_2, \hat{B}_3]] + [\hat{B}_2, [\hat{B}_3, \hat{B}_1]] + [\hat{B}_3, [\hat{B}_1, \hat{B}_2]] &= 0, \\
[\hat{B}_1, [\hat{B}_2, \hat{F}]] + [\hat{B}_2, [\hat{F}, \hat{B}_1]] + [\hat{F}, [\hat{B}_1, \hat{B}_2]] &= 0, \\
\{\hat{F}_1, [\hat{F}_2, \hat{B}]\} - \{\hat{F}_2, [\hat{B}, \hat{F}_1]\} + [\hat{B}, \{\hat{F}_1, \hat{F}_2\}] &= 0, \\
[\hat{F}_1, \{\hat{F}_2, \hat{F}_3\}] + [\hat{F}_2, \{\hat{F}_3, \hat{F}_1\}] + [\hat{F}_3, \{\hat{F}_1, \hat{F}_2\}] &= 0,
\end{aligned} \tag{5}$$

where ‘ $B$ ’ and ‘ $F$ ’ indicate the overall statistics of the operator involved.

5. Finally, an exercise in fermionic creation and annihilation operators and their anticommutation relations,

$$\{\hat{a}_\alpha, \hat{a}_\beta\} = \{\hat{a}_\alpha^\dagger, \hat{a}_\beta^\dagger\} = 0, \quad \{\hat{a}_\alpha, \hat{a}_\beta^\dagger\} = \delta_{\alpha, \beta}. \tag{6}$$

- (a) Calculate the commutators  $[\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger]$ ,  $[\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\delta]$  and  $[\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger \hat{a}_\delta]$ .
- (b) Consider two one-body operators  $\hat{A}_1$  and  $\hat{B}_1$  and let  $\hat{C}_1$  be their commutator,  $\hat{C}_1 = [\hat{A}_1, \hat{B}_1]$ . Let  $\hat{A}$  be the second-quantized forms of  $\hat{A}_{\text{tot}}$ ,

$$\hat{A} = \sum_{\alpha, \beta} \langle \alpha | \hat{A}_1 | \beta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta, \tag{7}$$

and ditto for the second-quantized  $\hat{B}$  and  $\hat{C}$ .

Verify that  $[\hat{A}, \hat{B}] = \hat{C}$ .

- (c) Calculate the commutator  $[\hat{a}_\mu^\dagger \hat{a}_\nu, \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta]$ .
- (d) The second quantized form of a two-body additive operator

$$\hat{B}_{\text{tot}} = \frac{1}{2} \sum_{i \neq j} \hat{B}_2(i^{\text{th}} \text{ and } j^{\text{th}} \text{ particles})$$

acting on identical fermions is

$$\hat{B} = \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha \otimes \beta | \hat{B}_2 | \gamma \otimes \delta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\delta \hat{a}_\gamma. \tag{8}$$

This expression is similar to its bosonic counterpart, but note the reversed order of the annihilation operators  $\hat{a}_\delta$  and  $\hat{a}_\gamma$ .

Consider a one-body operator  $\hat{A}_1$  and two two-body operators  $\hat{B}_2$  and  $\hat{C}_2$ . Show that if  $\hat{C}_2 = [(\hat{A}_1(1^{\text{st}}) + \hat{A}_1(2^{\text{nd}})), \hat{B}_2]$ , then the respective second-quantized operators in the fermionic Fock space satisfy  $\hat{C} = [\hat{A}, \hat{B}]$ .