

Question 1:

Let  $\Delta T^{\mu\nu} = \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu}$ . Regardless of the specific form of the  $\mathcal{K}^{[\lambda\mu]\nu}(\phi, \partial\phi)$  tensor, its anti-symmetry with respect to its first two indices implies

$$\partial_\mu \Delta T^{\mu\nu} = \partial_\mu \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu} = 0 \quad (\text{S.1})$$

and hence the first eq. (3). Furthermore,

$$\int d^3\mathbf{x} (\Delta T^{0\nu} = \partial_i \mathcal{K}^{i0\nu}) = \oint_{\substack{\text{boundary} \\ \text{of space}}} d^2\text{Area}_i \mathcal{K}^{i0\nu} \longrightarrow 0 \quad (\text{S.2})$$

when the integral is taken over the whole space, hence the second eq. (3).

Question 2:

In the Noether's formula (1) for the stress-energy tensor,  $\phi_a$  stand for the independent fields, however labeled. In the electromagnetic case, the independent fields are components of the 4-vector  $A_\lambda(x)$ , hence

$$\begin{aligned} T_{\text{Noether}}^{\mu\nu}(\text{EM}) &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - g^{\mu\nu} \mathcal{L} \\ &= -F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda}. \end{aligned} \quad (\text{S.3})$$

While the second term here is clearly both gauge invariant and symmetric in  $\mu \leftrightarrow \nu$ , the first term is neither.

Question 3:

Clearly, one can easily restore both symmetry and gauge invariance of the electromagnetic stress-energy tensor by replacing  $\partial^\nu A_\lambda$  in eq. (S.3) with  $F^\nu_\lambda$ , hence eq. (5). The correction

amounts to

$$T^{\mu\nu} - T_{\text{Noether}}^{\mu\nu} = -F^{\mu\lambda} (F_{\lambda}^{\nu} - \partial^{\nu}A_{\lambda} = -\partial_{\lambda}A^{\nu}) = \partial_{\lambda} (F^{\mu\lambda}A^{\nu}) - A^{\nu} (\partial_{\lambda}F^{\mu\lambda}) \quad (\text{S.4})$$

where the last term on the right hand side vanishes for the free electromagnetic field (which satisfies  $\partial_{\lambda}F^{\mu\lambda} = 0$ ). Consequently,

$$T^{\mu\nu} = T_{\text{Noether}}^{\mu\nu} + \mathcal{K}^{\lambda\mu\nu} \quad \text{where} \quad \mathcal{K}^{\lambda\mu\nu} = F^{\mu\lambda}A^{\nu} = -\mathcal{K}^{\mu\lambda\nu} \quad (\text{S.5})$$

in perfect agreement with eq. (2).

Question 4:

As explained in class, the Lagrangian  $\mathcal{L} = -\frac{1}{4}F_{\kappa\lambda}F^{\kappa\lambda} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)$ . Combining this fact with eq. (5), we have the energy density

$$\mathcal{H} = T^{00} = -F^{0i}F_i^0 - \mathcal{L} = +\mathbf{E}^2 - \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) \quad (\text{S.6})$$

in agreement with the standard electromagnetic formulæ (note the  $c = 1$ , *rationalized* units here). Likewise, the energy flux and the momentum density are

$$S^i = T^{i0} = T^{0i} = -F^{0j}F_j^i = -(-E^j)(\epsilon^{ijk}B^k) = +\epsilon^{ijk}E^jB^k = (\mathbf{E} \times \mathbf{B})^i, \quad (\text{S.7})$$

in agreement with the Poynting vector  $\mathbf{S} = \mathbf{E} \times \mathbf{B}$  (again, in the  $c = 1$ , *rationalized* units). Finally, the (3-dimensional) stress tensor is

$$\begin{aligned} T_{\text{EM}}^{ij} &= -F^{i\lambda}F_{\lambda}^j - g^{ij}\mathcal{L} = -F^{i0}F_0^j - F^{ik}F_k^j + \delta^{ij}\mathcal{L} \\ &= -E^iE^j + \epsilon^{ik\ell}B^{\ell}\epsilon^{jkm}B^m + \frac{1}{2}\delta^{ij}(\mathbf{E}^2 - \mathbf{B}^2) \\ &= -E^iE^j - B^iB^j + \frac{1}{2}\delta^{ij}(\mathbf{E}^2 + \mathbf{B}^2). \end{aligned} \quad (\text{S.8})$$

Question 5:

In a sense, eq. (6) follows from eq. (S.4), but it is just as easy to derive it directly from Maxwell

equations. Starting with eq. (5), we immediately have

$$\partial_\mu T_{\text{EM}}^{\mu\nu} = -(\partial_\mu F^{\mu\lambda})F_\lambda^\nu - F^{\mu\lambda}(\partial_\mu F_\lambda^\nu) + \frac{1}{2}F_{\kappa\lambda}(\partial^\nu F^{\kappa\lambda}). \quad (\text{S.9})$$

Using the antisymmetry  $F^{\mu\lambda} = -F^{\lambda\mu}$ , we rewrite the second term on the right hand side as

$$-F^{\mu\lambda}\partial_\mu F_\lambda^\nu = +F_{\mu\lambda}\partial^\mu F^{\lambda\nu} = +F_{\mu\lambda}\partial^\lambda F^{\nu\mu} = \frac{1}{2}F_{\mu\lambda}(\partial^\lambda F^{\nu\mu} + \partial^\mu F^{\lambda\nu}) = -\frac{1}{2}F_{\mu\lambda}(\partial^\nu F^{\mu\lambda}) \quad (\text{S.10})$$

(using Maxwell equation  $\partial^\lambda F^{\nu\mu} + \partial^\mu F^{\lambda\nu} + \partial^\nu F^{\mu\lambda} = 0$ ) and thus precisely cancels the third term on the right hand side of eq. (S.9). Consequently,

$$\partial_\mu T_{\text{EM}}^{\mu\nu} = -(\partial_\mu F^{\mu\lambda})F_\lambda^\nu = -J^\lambda F_\lambda^\nu \quad (\text{S.11})$$

by the other Maxwell equation  $\partial_\mu F^{\mu\lambda} = J^\lambda$ . *Q.E.D.*

#### Question 6:

From the spacetime point of view, a covariant derivative acting upon a field  $\Phi_q$  of charge  $q$  is a differential operator  $D_\mu = \partial_\mu + iqA_\mu(x)$ . Consequently,

$$\begin{aligned} [D_\mu, D_\nu] &= [\partial_\mu, \partial_\nu] + iq[\partial_\mu, A_\nu(x)] + iq[A_\mu(x), \partial_\nu] - q^2[A_\mu(x), A_\nu(x)] \\ &= 0 + iq(\partial_\mu A_\nu(x)) + iq(-\partial_\nu A_\mu(x)) - q^2(0) \\ &= iq(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)) = iqF_{\mu\nu}(x). \end{aligned} \quad (\text{S.12})$$

*Q.E.D.*

#### Question 7:

The covariant derivatives in the Lagrangian (9) hide the  $A_\mu$  dependence:

$$\frac{\partial D_\nu \Phi}{\partial A_\mu} = iq\Phi\delta_\nu^\mu \quad \text{and} \quad \frac{\partial D_\nu \Phi^*}{\partial A_\mu} = -iq\Phi^*\delta_\nu^\mu, \quad (\text{S.13})$$

and therefore

$$J^\mu = -\frac{\partial \mathcal{L}}{\partial A_\mu} = -iq\Phi \frac{\partial \mathcal{L}}{\partial D_\mu \Phi} + -iq\Phi^* \frac{\partial \mathcal{L}}{\partial D_\mu \Phi^*} = -iq\Phi D^\mu \Phi^* + iq\Phi^* D^\mu \Phi. \quad (\text{S.14})$$

Thanks to the covariance of the derivative  $D^\mu$  this current is gauge invariant:

$$\left. \begin{aligned} \Phi(x) &\rightarrow e^{+iq\alpha(x)}\Phi(x), & D^\mu\Phi(x) &\rightarrow e^{+iq\alpha(x)}D^\mu\Phi(x), \\ \Phi^*(x) &\rightarrow e^{-iq\alpha(x)}\Phi^*(x), & D^\mu\Phi^*(x) &\rightarrow e^{-iq\alpha(x)}D^\mu\Phi^*(x) \end{aligned} \right\} \implies J^\mu(x) \rightarrow J^\mu(x). \quad (\text{S.15})$$

Question 8:

As discussed in class, the equations of motions for the scalar fields  $\Phi$  and  $\Phi^*$  can be written in the gauge invariant form as

$$D_\mu D^\mu \Phi = -m^2 \Phi, \quad D_\mu D^\mu \Phi^* = -m^2 \Phi^*. \quad (\text{S.16})$$

Consequently,

$$\Phi^* D_\mu D^\mu \Phi - \Phi D_\mu D^\mu \Phi^* = 0, \quad (\text{S.17})$$

which in turn leads to conservation of the electric current (S.14):

$$\begin{aligned} \partial_\mu(\Phi^* D^\mu \Phi) &= D_\mu(\Phi^* D^\mu \Phi) = (D_\mu \Phi^*)(D^\mu \Phi) + \Phi^* (D_\mu D^\mu \Phi), \\ \partial_\mu(\Phi D^\mu \Phi^*) &= D_\mu(\Phi D^\mu \Phi^*) = (D_\mu \Phi)(D^\mu \Phi^*) + \Phi (D_\mu D^\mu \Phi^*), \end{aligned} \quad (\text{S.18})$$

and therefore

$$\partial_\mu J^\mu = -iq\partial_\mu(\Phi^* D^\mu \Phi) + iq\partial_\mu(\Phi D^\mu \Phi^*) = -iq\Phi^* D_\mu D^\mu \Phi + iq\Phi D_\mu D^\mu \Phi^* = 0. \quad (\text{S.19})$$

Question 9:

According to the Noether theorem (1),

$$\begin{aligned} T_{\text{Noether}}^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial^\nu \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^*)} \partial^\nu \Phi^* - g^{\mu\nu} \mathcal{L} \\ &= T_{\text{Noether}}^{\mu\nu}(\text{EM}) + T_{\text{Noether}}^{\mu\nu}(\text{mat}) \end{aligned} \quad (\text{S.20})$$

where  $T_{\text{Noether}}^{\mu\nu}(\text{EM})$  is exactly as in eq. (S.3) for free EM fields and

$$T_{\text{Noether}}^{\mu\nu}(\text{mat}) = D^\mu \Phi^* \partial^\nu \Phi + D^\mu \Phi \partial^\nu \Phi^* - g^{\mu\nu} (D^\lambda \Phi^* D_\lambda \Phi - m^2 \Phi^* \Phi). \quad (\text{S.21})$$

Both terms on the second line of eq. (S.20) lack  $\mu \leftrightarrow \nu$  symmetry and gauge invariance and thus need corrections à la (2). In fact, the same  $\mathcal{K}^{[\lambda\mu]\nu} = F^{\mu\lambda} A^\nu$  we used to improve the free

electromagnetic stress-energy tensor will now improve both the  $T_{\text{EM}}^{\mu\nu}$  and  $T_{\text{mat}}^{\mu\nu}$  at the same time! Indeed, according to eq. (S.4),

$$\partial_\lambda \left( \mathcal{K}^{[\lambda\mu]\nu} \stackrel{\text{def}}{=} F^{\mu\lambda} A^\nu \right) = T^{\mu\nu}(\text{EM}) - T_{\text{Noether}}^{\mu\nu}(\text{EM}) + A^\nu \left( \partial_\lambda F^{\mu\lambda} \right), \quad (\text{S.22})$$

where the last term on the right hand side is precisely the difference between the improved stress-energy tensor (12) for the charged fields and the Noether tensor (S.21) for the same.

The proof of the last assertion is based upon Maxwell equation  $\partial_\lambda F^{\lambda\mu} = J^\mu$  and the current (S.14), thus

$$\partial_\lambda F^{\mu\lambda} = -J^\mu = -iq\Phi^* D^\mu \Phi + iq\Phi D^\mu \Phi^*$$

and therefore

$$\begin{aligned} A^\nu \left( \partial_\lambda F^{\lambda\mu} \right) &= (-iqA^\nu \Phi^*) D^\mu \Phi + (+iqA^\nu \Phi) D^\mu \Phi^* \\ &= D^\mu \Phi (D^\nu \Phi^* - \partial^\nu \Phi^*) + D^\mu \Phi^* (D^\nu \Phi - \partial^\nu \Phi) \\ &= T^{\mu\nu}(\text{mat}) - T_{\text{Noether}}^{\mu\nu}(\text{mat}). \end{aligned} \quad (\text{S.23})$$

*Q.E.D.*

#### Question 10:

Because the fields  $\Phi(x)$  and  $\Phi^*(x)$  have opposite electric charges, their product is neutral and therefore  $\partial_\mu(\Phi^*\Phi) = D_\mu(\Phi^*\Phi) = (D_\mu\Phi^*)\Phi + \Phi^*(D_\mu\Phi)$ . Similarly,

$$\begin{aligned} \partial_\mu ((D^\mu\Phi^*) (D^\nu\Phi)) &= (D_\mu D^\mu\Phi^*) (D^\nu\Phi) + (D^\mu\Phi^*) (D_\mu D^\nu\Phi) \\ &= -m^2\Phi^* (D^\nu\Phi) + (D_\mu\Phi^*) (D^\nu D^\mu\Phi + iqF^{\mu\nu}\Phi) \end{aligned} \quad (\text{S.24})$$

where we have applied the field equation  $(D_\mu D^\mu + m^2)\Phi^*(x) = 0$  to the first term on the right hand side and used eq. (8) to expand the second term. Likewise,

$$\begin{aligned} \partial_\mu ((D^\mu\Phi) (D^\nu\Phi^*)) &= (D_\mu D^\mu\Phi) (D^\nu\Phi^*) + (D^\mu\Phi) (D_\mu D^\nu\Phi^*) \\ &= -m^2\Phi (D^\nu\Phi^*) + (D_\mu\Phi) (D^\nu D^\mu\Phi^* - iqF^{\mu\nu}\Phi^*), \end{aligned} \quad (\text{S.25})$$

and finally

$$\begin{aligned}
\partial_\mu \left[ -g^{\mu\nu} \left( D_\lambda \Phi^* D^\lambda \Phi - m^2 \Phi^* \Phi \right) \right] &= -\partial^\nu \left( D_\lambda \Phi^* D^\lambda \Phi \right) + m^2 \partial^\nu \left( \Phi^* \Phi \right) \\
&= -\left( D^\nu D^\mu \Phi^* \right) \left( D_\mu \Phi \right) - \left( D_\mu \Phi^* \right) \left( D^\nu D^\mu \Phi \right) \quad (\text{S.26}) \\
&\quad + m^2 \Phi \left( D^\nu \Phi^* \right) + m^2 \Phi^* \left( D^\nu \Phi \right).
\end{aligned}$$

Together, the left hand sides of eqs. (S.24), (S.25) and (S.26) comprise  $\partial_\mu T_{\text{mat}}^{\mu\nu}$  — *cf.* eq. (12). On the other hand, combining the right hand sides of these three equations results in massive cancellation of all terms except those containing the gauge field strength tensor  $F^{\mu\nu}$ . Hence,

$$\partial_\mu T_{\text{mat}}^{\mu\nu} = iqF^{\mu\nu} \left( \Phi D_\mu \Phi^* - \Phi^* D_\mu \Phi \right) = F^{\mu\nu} J_\nu. \quad (\text{S.27})$$

*Q.E.D.*