Problem **1**(a):

As discussed in class for the massless case (EM), $\partial\mathcal{L}/\partial(\partial_\mu A_\nu) = -F^{\mu\nu}$.

Clearly, $\partial\mathcal{L}/\partial(A_\nu) = +m^2 A^\nu - J^\nu$. Hence, the Euler–Lagrange field equation is

$$-\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu \mathcal{A}_\nu)} + \frac{\partial\mathcal{L}}{\partial(\mathcal{A}_\nu)} \equiv \partial_\mu F^{\mu\nu} + m^2 A^\nu - J^\nu = 0, \tag{S.1}$$

or in terms of A^ν and their explicit derivatives,

$$\partial^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) + m^2 A^\nu - J^\nu = 0. \tag{S.2}$$

Problem **1**(b):

Take the divergence ∂_ν of the field equation (S.2); the first two terms cancel out while the rest becomes

$$m^2\,\partial_\nu A^\nu - \partial_\nu J^\nu = 0. \tag{S.3}$$

In the massless case, this equation enforces the current conservation $\partial_\nu J^\nu = 0$ regardless of the 4–vector potential $A^\nu(x)$, but there is no such constraint in the massive case at hand. Instead, eq. (S.3) simply relates the current divergence to the 4–potential divergence. In particular, *if the current happens to satisfy $\partial_\nu J^\nu$, then — and only then — eq.* (S.3) *requires $\partial_\nu A^\nu = 0$ as well.* Consequently, the field equation (S.2) simplifies to $(\partial^2 + m^2)A^\nu = J^\nu$. $\mathcal{Q.E.D.}$

Problem **1**(c):

As in (a), $\partial\mathcal{L}/\partial(\partial^\mu A^\nu) = -F_{\mu\nu}$.

In particular, $\partial\mathcal{L}/\partial(\partial^0 A^i) = -F_{0i} = -E^i$ while $\partial\mathcal{L}/\partial(\partial^0 A^0) = -F_{00} = 0$.

Problem $\mathbf{1}$(d):

In terms of the Hamiltonian and Lagrangian densities, eq. (3) means

$$\mathcal{H} \;=\; -\dot{\mathbf{A}}\cdot\mathbf{E} \;-\; \mathcal{L}. \tag{3$'$}$$

In terms of $\mathbf{A}$, $\mathbf{E}$ and A_0,

$$\begin{aligned} \dot{\mathbf{A}} &= -\mathbf{E} \;-\; \nabla A_0, \\ -\dot{\mathbf{A}}\cdot\mathbf{E} &= \mathbf{E}^2 \;+\; \mathbf{E}\cdot\nabla A_0, \\ \mathcal{L} &= \tfrac{1}{2}\left(\mathbf{E}^2 - (\nabla\times\mathbf{A})^2\right) + \;\tfrac{1}{2}m^2(A_0^2 - \mathbf{A}^2) \;-\; (A_0J_0 - \mathbf{A}\cdot\mathbf{J}). \end{aligned} \tag{S.4}$$

Consequently,

$$\mathcal{H} \;= \tfrac{1}{2}\mathbf{E}^2 \;+\; \mathbf{E}\cdot\nabla A_0 \;-\; \tfrac{1}{2}m^2A_0^2 \;+\; A_0J_0 \;+\; \tfrac{1}{2}(\nabla\times\mathbf{A})^2 \;+\; \tfrac{1}{2}m^2\mathbf{A}^2 \;-\; \mathbf{A}\cdot\mathbf{J}, \tag{S.5}$$

which immediately leads to eq. (4) (after one integration by parts). $\mathcal{Q.E.D.}$

Problem $\mathbf{1}$(e):

Expanding the left hand side of eq. (5) gives us

$$\frac{\delta H}{\delta A_0(\mathbf{x})} \;\equiv\; \frac{\partial\mathcal{H}}{\partial(A_0)} \;-\; \nabla_i\frac{\partial\mathcal{H}}{\partial(\nabla_i A_0)} \;=\; -m^2A_0 \;+\; J_0 \;-\; \nabla_i E^i,$$

hence

$$A_0(\mathbf{x},t) \;=\; \frac{J_0 \;-\; \nabla\cdot\mathbf{E}}{m^2} \tag{S.6}$$

everywhere in spacetime. Please note that there are no derivatives at the left hand side of this formula.

Likewise, expanding the right hand side of the first eq. (6) yields

$$\frac{\delta H}{\delta E^i(\mathbf{x})} \;\equiv\; \frac{\partial\mathcal{H}}{\partial(E^i)} \;-\; \nabla_j\frac{\partial\mathcal{H}}{\partial(\nabla_j E^i)} \;=\; E^i \;+\; \nabla_i A_0$$

and hence

$$\frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) \;=\; -\mathbf{E} \;-\; \nabla A_0. \tag{S.7}$$

Similarly,

$$\frac{\delta H}{\delta A^i(\mathbf{x})} \;\equiv\; \frac{\partial\mathcal{H}}{\partial(\mathcal{A}\rangle)} \;-\; \nabla_j\frac{\partial\mathcal{H}}{\partial(\nabla_j A^i)} \;=\; m^2 A^i \;-\; J^i \;-\; \nabla_j(\epsilon^{jik}(\nabla\times\mathbf{A})^k)$$

and hence

$$\frac{\partial}{\partial t}\mathbf{E}(\mathbf{x},t) \;=\; m^2\mathbf{A} \;-\; \mathbf{J} \;+\; \nabla\times(\nabla\times\mathbf{A}). \tag{S.8}$$

Problem **1**(f):

In the 3–space notations, the Euler–Lagrange field equations (S.1) become

$$\nabla\cdot\mathbf{E} \;-\; m^2 A_0 \;=\; J_0\,, \tag{S.9}$$

$$-\dot{\mathbf{E}} \;+\; \nabla\times\mathbf{B} \;+\; m^2\mathbf{A} \;=\; \mathbf{J}, \tag{S.10}$$

where

$$\mathbf{E} \;\stackrel{\text{def}}{=}\; -\dot{\mathbf{A}} \;-\; \nabla A_0, \tag{S.11}$$

$$\mathbf{B} \;\stackrel{\text{def}}{=}\; \nabla\times\mathbf{A}. \tag{S.12}$$

Clearly, eq. (S.9) is equivalent to eq. (S.6) while eq. (S.10) is equivalent to eq. (S.8) (provided **B** is defined as in eq. (S.12)). Finally, eq. (S.11) is equivalent to eq. (S.7), although their origins differ: In the Lagrangian formalism, eq. (S.11) is the definition of the **E** field in terms of A_0, **A** and their derivatives, while in the Hamiltonian formalism, **E** is an independent conjugate field and eq. (S.7) is the dynamical equation of motion for the $\dot{\mathbf{A}}$. $\mathcal{Q.E.D.}$

Problem **2**:

Assume generic commutation relations

$$\begin{aligned}
\left[\hat{E}^j(\mathbf{x},t),\hat{E}^k(\mathbf{x}',t'=t)\right] &= \hat{f}^{jk}(\mathbf{x}-\mathbf{x}'),\\
\left[\hat{B}^j(\mathbf{x},t),\hat{B}^k(\mathbf{x}',t'=t)\right] &= \hat{g}^{jk}(\mathbf{x}-\mathbf{x}'),\\
\left[\hat{E}^j(\mathbf{x},t),\hat{B}^k(\mathbf{x}',t'=t)\right] &= \hat{h}^{jk}(\mathbf{x}-\mathbf{x}').
\end{aligned} \tag{S.13}$$

Commuting the quantum electromagnetic fields with the Hamiltonian (9), we obtain

$$\begin{aligned}
i\frac{\partial}{\partial t}\hat{E}^j(\mathbf{x},t) &= \left[\hat{E}^j(\mathbf{x},t),\hat{H}(t)\right]\\
&= \int d^3\mathbf{x}'\left(\tfrac{1}{2}\left\{\hat{f}^{jk}(\mathbf{x}-\mathbf{x}'),\hat{E}^k(\mathbf{x}',t)\right\} + \tfrac{1}{2}\left\{\hat{h}^{jk}(\mathbf{x}-\mathbf{x}'),\hat{B}^k(\mathbf{x}',t)\right\}\right)\\
i\frac{\partial}{\partial t}\hat{B}^j(\mathbf{x},t) &= \left[\hat{B}^j(\mathbf{x},t),\hat{H}(t)\right]\\
&= \int d^3\mathbf{x}'\left(\tfrac{1}{2}\left\{\hat{g}^{jk}(\mathbf{x}-\mathbf{x}'),\hat{B}^k(\mathbf{x}',t)\right\} - \tfrac{1}{2}\left\{\hat{h}^{kj}(\mathbf{x}'-\mathbf{x}),\hat{E}^k(\mathbf{x}',t)\right\}\right)
\end{aligned} \tag{S.14}$$

Comparing these Heisenberg equations with the time-dependent Maxwell equations (8), we see that we want

$$\begin{aligned}
\tfrac{1}{2}\left\{\hat{f}^{jk}(\mathbf{x}-\mathbf{x}'),\hat{E}^k(\mathbf{x}',t)\right\} &= 0,\\
\tfrac{1}{2}\left\{\hat{h}^{jk}(\mathbf{x}-\mathbf{x}'),\hat{B}^k(\mathbf{x}',t)\right\} &= i\epsilon^{j\ell k}\frac{\partial}{\partial x^\ell}\delta^{(3)}(\mathbf{x}-\mathbf{x}')\,\hat{B}^k(\mathbf{x}',t),\\
\tfrac{1}{2}\left\{\hat{g}^{jk}(\mathbf{x}-\mathbf{x}'),\hat{B}^k(\mathbf{x}',t)\right\} &= 0,\\
\tfrac{1}{2}\left\{\hat{h}^{kj}(\mathbf{x}'-\mathbf{x}),\hat{E}^k(\mathbf{x}',t)\right\} &= i\epsilon^{j\ell k}\frac{\partial}{\partial x^\ell}\delta^{(3)}(\mathbf{x}-\mathbf{x}')\,\hat{E}^k(\mathbf{x}',t),
\end{aligned} \tag{S.15}$$

and the simplest way to satisfy these requirements is to have c–number valued (rather than operator-valued) functions

$$\begin{aligned}
f^{jk}(\mathbf{x}-\mathbf{x}') \equiv g^{jk}(\mathbf{x}-\mathbf{x}') &\equiv 0,\\
h^{jk}(\mathbf{x}-\mathbf{x}') = h^{kj}(\mathbf{x}'-\mathbf{x}) &= i\epsilon^{j\ell k}\frac{\partial}{\partial x^\ell}\delta^{(3)}(\mathbf{x}-\mathbf{x}').
\end{aligned} \tag{S.16}$$

In other words

$$\begin{aligned}
\left[\hat{E}^j(\mathbf{x},t), \hat{E}^k(\mathbf{x}',t'=t)\right] &= 0,\\
\left[\hat{B}^j(\mathbf{x},t), \hat{B}^k(\mathbf{x}',t'=t)\right] &= 0,\\
\left[\hat{E}^j(\mathbf{x},t), \hat{B}^k(\mathbf{x}',t'=t)\right] &= i\epsilon^{j\ell k}\,\frac{\partial}{\partial x^\ell}\,\delta^{(3)}(\mathbf{x}-\mathbf{x}') \;=\; i\epsilon^{k\ell j}\,\frac{\partial}{\partial x'^\ell}\,\delta^{(3)}(\mathbf{x}'-\mathbf{x}).
\end{aligned} \tag{S.17}$$

It remains to verify the consistency of these commutation relation with the transversality of the free electromagnetic fields: $\nabla\cdot\hat{\mathbf{E}}$ and $\nabla\cdot\hat{\mathbf{B}}$ vanish (*cf.* eq. (7)) and thus have to commute with everything. On the other hand, the commutation relations of the $\nabla\cdot\hat{\mathbf{E}}$ and $\nabla\cdot\hat{\mathbf{B}}$ follow from those of the individual components of the fields:

$$\left[\nabla\cdot\hat{\mathbf{E}}(\mathbf{x}), \text{whatever}(\mathbf{x}')\right] \;=\; \frac{\partial}{\partial x^j}\left[\hat{E}^j(\mathbf{x}), \text{same whatever}(\mathbf{x}')\right] \tag{S.18}$$

and ditto for the magnetic field. Applying this rule to the commutation relations (S.17) (and making use of $\epsilon^{j\ell k}\,\nabla_j\nabla_\ell = 0$), we have

$$\begin{aligned}
\left[\nabla\cdot\hat{\mathbf{E}}(\mathbf{x}), \hat{E}^k(\mathbf{x}')\right] &= 0,\\
\left[\nabla\cdot\hat{\mathbf{E}}(\mathbf{x}), \hat{B}^k(\mathbf{x}')\right] &= i\epsilon^{j\ell k}\,\nabla_j\nabla_\ell\,\delta^{(3)}(\mathbf{x}-\mathbf{x}') \;=\; 0,\\
\left[\nabla\cdot\hat{\mathbf{B}}(\mathbf{x}), \hat{B}^k(\mathbf{x}')\right] &= 0,\\
\left[\nabla\cdot\hat{\mathbf{B}}(\mathbf{x}), \hat{E}^k(\mathbf{x}')\right] &= -i\epsilon^{j\ell k}\,\nabla_j\nabla_\ell\,\delta^{(3)}(\mathbf{x}-\mathbf{x}') \;=\; 0.
\end{aligned} \tag{S.19}$$

$\mathcal{Q.E.D.}$

Problem **3**(a):

This is a simple exercise of the Leibniz rule for commutators:

$$\begin{aligned}
[\hat{a}^\dagger_\alpha\hat{a}_\beta, \hat{a}^\dagger_\gamma] &= [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\gamma]\hat{a}_\beta \;+\; \hat{a}^\dagger_\alpha[\hat{a}_\beta, \hat{a}^\dagger_\gamma] \;=\; 0 \;+\; \hat{a}^\dagger_\alpha\delta_{\beta,\gamma} \;=\; \delta_{\beta,\gamma}\hat{a}^\dagger_\alpha\,,\\
[\hat{a}^\dagger_\alpha\hat{a}_\beta, \hat{a}_\delta] &= [\hat{a}^\dagger_\alpha, \hat{a}_\delta]\hat{a}_\beta \;+\; \hat{a}^\dagger_\alpha[\hat{a}_\beta, \hat{a}_\delta] \;=\; -\delta_{\alpha,\delta}\hat{a}_\beta \;+\; 0 \;=\; -\delta_{\alpha,\delta}\hat{a}_\beta\,,\\
[\hat{a}^\dagger_\alpha\hat{a}_\beta, \hat{a}^\dagger_\gamma\hat{a}_\delta] &= [\hat{a}^\dagger_\alpha\hat{a}_\beta, \hat{a}^\dagger_\gamma]\hat{a}_\delta \;+\; \hat{a}^\dagger_\gamma[\hat{a}^\dagger_\alpha\hat{a}_\beta, \hat{a}_\gamma\hat{a}_\delta] \;=\; \delta_{\beta,\gamma}\hat{a}^\dagger_\alpha\hat{a}_\delta \;-\; \delta_{\alpha,\delta}\hat{a}^\dagger_\gamma\hat{a}_\beta\,.
\end{aligned} \tag{S.20}$$

Problem **3**(b):

Given

$$\hat{A} \;=\sum_{\alpha,\beta}\langle\alpha|\,\hat{A}_1\,|\beta\rangle\;\hat{a}^\dagger_\alpha\hat{a}_\beta \qquad \text{and} \quad \hat{B} \;=\sum_{\gamma,\delta}\langle\gamma|\,\hat{B}_1\,|\delta\rangle\;\hat{a}^\dagger_\gamma\hat{a}_\delta\,,$$

we immediately have

$$\begin{aligned}
[\hat{A},\hat{B}] \;&=\sum_{\alpha,\beta,\gamma,\delta}\langle\alpha|\,\hat{A}_1\,|\beta\rangle\;\langle\gamma|\,\hat{B}_1\,|\delta\rangle\;[\hat{a}^\dagger_\alpha\hat{a}_\beta,\hat{a}^\dagger_\gamma\hat{a}_\delta]\\
&\langle\!\langle\text{using (S.20)}\rangle\!\rangle\\
&=\sum_{\alpha,\beta,\gamma,\delta}\langle\alpha|\,\hat{A}_1\,|\beta\rangle\;\langle\gamma|\,\hat{B}_1\,|\delta\rangle\;\Big(\delta_{\beta,\gamma}\hat{a}^\dagger_\alpha\hat{a}_\delta\;-\;\delta_{\alpha,\delta}\hat{a}^\dagger_\gamma\hat{a}_\beta\Big)\\
&=\sum_{\alpha,\delta}\hat{a}^\dagger_\alpha\hat{a}_\delta\times\sum_{\beta=\gamma}\langle\alpha|\,\hat{A}_1\,|\gamma\rangle\,\langle\gamma|\,\hat{B}_1\,|\delta\rangle\;-\sum_{\beta,\gamma}\hat{a}^\dagger_\gamma\hat{a}_\beta\times\sum_{\alpha=\delta}\langle\gamma|\,\hat{B}_1\,|\alpha\rangle\,\langle\alpha|\,\hat{A}_1\,|\beta\rangle\\
&=\sum_{\alpha,\delta}\hat{a}^\dagger_\alpha\hat{a}_\delta\;\langle\alpha|\,\hat{A}_1\hat{B}_1\,|\delta\rangle\;-\sum_{\beta,\gamma}\hat{a}^\dagger_\gamma\hat{a}_\beta\;\langle\gamma|\,\hat{B}_1\hat{A}_1\,|\beta\rangle\\
&\langle\!\langle\text{renaming summation indices}\rangle\!\rangle\\
&=\sum_{\alpha,\beta}\hat{a}^\dagger_\alpha\hat{a}_\beta\times\Big(\langle\alpha|\,\hat{A}_1\hat{B}_1\,|\beta\rangle\;-\;\langle\alpha|\,\hat{B}_1\hat{A}_1\,|\beta\rangle\Big)\\
&=\sum_{\alpha,\beta}\hat{a}^\dagger_\alpha\hat{a}_\beta\times\langle\alpha|\Big([\hat{A}_1,\hat{B}_1]=\hat{C}_1\Big)\,|\beta\rangle\;\equiv\;\hat{C}.
\end{aligned} \tag{S.21}$$