Problem $\mathbf{1}$(a):

First, let us verify eq. (3) for a wave function $\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ of the form (1), that is, for the quantum state $|N,\Psi\rangle = |\alpha_1,\ldots,\alpha_N\rangle = \big|\{n_\beta\}\big\rangle$. Note that the number of distinct permutations of N one-particle modes $\alpha_1,\ldots,\alpha_N$ corresponding to occupation numbers n_β is precisely $\big(N!\,/\prod_\beta n_\beta!\big)$, so the wave function (1) is properly normalized.

Let us define $\Psi'(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})$ according to eq. (3). Using orthonormality of the 1-particle wave functions $\phi_\beta(\mathbf{x})$, we have

$$\begin{aligned}\Psi'(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) &= \sqrt{\frac{\prod_\beta n_\beta!}{N-1!}}\int d^3\mathbf{x}_N\,\phi^*_\gamma(\mathbf{x}_N)\sum_{(\tilde\alpha_1,\ldots,\tilde\alpha_N)}\phi_{\tilde\alpha_1}(\mathbf{x}_1)\cdots\phi_{\tilde\alpha_N}(\mathbf{x}_N)\\ &= \sqrt{\frac{\prod_\beta n_\beta!}{(N-1)!}}\sum_{(\tilde\alpha_1,\ldots,\tilde\alpha_N)}\phi_{\tilde\alpha_1}(\mathbf{x}_1)\cdots\phi_{\tilde\alpha_{N-1}}(\mathbf{x}_{N-1})\times\delta_{\tilde\alpha_N,\gamma}\,,\end{aligned}\tag{S.1}$$

which leads to two distinct situations: (A) If $n_\gamma=0$ *i.e.*, if none of the $\alpha_1,\ldots,\alpha_N$ equals γ, then for any permutation $(\tilde\alpha_1,\ldots,\tilde\alpha_N)$ $\tilde\alpha_n\neq\gamma$ and every term in the sum (S.1) vanishes, thus $\Psi'(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})\equiv 0$. Clearly, this agrees with $\hat a_\gamma\,\big|\{n_\beta\}\big\rangle=0$ for $n_\gamma=0$. (B) On the other hand, if $n_\gamma>0$ then without loss of generality we let $\alpha_N=\gamma$ (remember that the list $(\alpha_1,\ldots,\alpha_N)$ id un-ordered). Consequently, the $\delta_{\tilde\alpha_N,\gamma}$ factor in eq. (S.1) restricts permutations $(\tilde\alpha_1,\ldots,\tilde\alpha_N)$ to $\tilde\alpha_N=\gamma=\alpha_N$, and since we count distinct permutations only, we have

$$\Psi'(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) = \sqrt{\frac{\prod_\beta n_\beta!}{N-1!}}\sum_{\substack{\text{distinct permutations}\\ (\tilde\alpha_1,\ldots,\tilde\alpha_{N-1})\text{ of }(\alpha_1,\ldots,\alpha_{N-1})}}\phi_{\tilde\alpha_1}(\mathbf{x}_1)\cdots\phi_{\tilde\alpha_{N-1}}(\mathbf{x}_{N-1}).\tag{S.2}$$

Comparing this expression to eq. (1), we see that Ψ' is the wave function of the state $|\alpha_1,\ldots,\alpha_{N-1}\rangle = \Big|\{n'_\beta=n_\beta-\delta_{\beta,\gamma}\}\Big\rangle$, except for the normalization factor

$$\sqrt{\frac{\prod_\beta n_\beta!}{N-1!}} = \sqrt{n_\gamma}\times\sqrt{\frac{\prod_\beta n'_\beta!}{N-1!}}\quad\text{instead of}\quad\sqrt{\frac{\prod_\beta n'_\beta!}{N-1!}}\,.\tag{S.3}$$

In other words,

$$\big|(N-1),\Psi'\big\rangle \;=\; \sqrt{n_\gamma}\,|\alpha_1,\ldots,\alpha_{N-1}\rangle \;=\; \hat{a}_\gamma\,|\alpha_1,\ldots,\alpha_N\rangle\,, \qquad \text{for } \alpha_N=\gamma, \tag{S.4}$$

and we already saw that for $\alpha_1,\ldots,\alpha_N\neq\gamma$, $|\Psi'\rangle = 0 = \hat{a}\,|\alpha_1,\ldots,\alpha_N\rangle$ as well. Thus,

$$\forall\,|\Psi\rangle = |\alpha_1,\ldots,\alpha_N\rangle:\ \big|\Psi'\big\rangle \;=\; \hat{a}_\gamma\,|\Psi\rangle\,, \tag{S.5}$$

and by linearity of eq. (3) it follows that $|\Psi'\rangle = \hat{a}_\gamma\,|\Psi\rangle$ for any linear combination of the $|\alpha_1,\ldots,\alpha_N\rangle$ states. As we saw in class, such states form a complete basis of the N-boson Hilbert space, therefore

$$\big|\Psi'\big\rangle \;=\; \hat{a}_\gamma\,|\Psi\rangle \quad \forall\,|\Psi\rangle\in\mathcal{H}_{N\,\text{bosons}}\,. \qquad \mathcal{Q.E.D.}$$

Problem **1**(b):

First, consider a one-body operator of the form $\hat{O} = |\alpha\rangle\langle\beta|$. For this operator, the second-quantized $\hat{O}^{(2)}_{\text{tot}}$ is simply $\hat{a}^\dagger_\alpha\hat{a}_\beta$ while the first-quantized $\hat{O}^{(1)}_{\text{tot}}$ has matrix elements

$$\begin{aligned}
&\langle N,\Psi_1|\,\hat{O}^{(1)}_{\text{tot}}\,|N,\Psi_2\rangle \;= \\
&\quad= \sum_{i=1}^{N}\int d^3\mathbf{x}_1\cdots\int d^3\mathbf{x}_N\int d^3\mathbf{x}'_i\,\Psi_1^*(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_N)\phi_\alpha(\mathbf{x}_i)\phi_\beta^*(\mathbf{x}'_i)\Psi_2(\mathbf{x}_1,\ldots,\mathbf{x}'_i,\ldots,\mathbf{x}_N) \\
&\quad= N\int d^3\mathbf{x}_1\cdots\int d^3\mathbf{x}_N\int d^3\mathbf{x}'_N\,\Psi_1^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N)\phi_\alpha(\mathbf{x}_N)\phi_\beta^*(\mathbf{x}'_N)\Psi_2(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}'_N) \\
&\quad= \int d^3\mathbf{x}_1\cdots\int d^3\mathbf{x}_{N-1}\,\Psi_1'^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})\,\Psi_2'(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})
\end{aligned}$$

where the second equality follows from the total symmetry of the wave functions Ψ_1 and Ψ_2, and the $(N-1)$-particle wave functions $\Psi'_{1,2}(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})$ on the last line are defined according to eq. (3) (but using $\hat{a}_\alpha$ instead of $\hat{a}_\gamma$ for the Ψ'_1 and $\hat{a}_\beta$ for the Ψ_2). In light of problem (a), this means

$$\begin{aligned}
\langle N,\Psi_1|\,\hat{O}^{(1)}_{\text{tot}}\,|N,\Psi_2\rangle \;=\; \big\langle(N-1),\Psi'_1\big|(N-1),\Psi'_2\big\rangle \;&=\; \langle N,\Psi_1|\,\hat{a}^\dagger_\alpha\hat{a}_\beta\,|N,\Psi_2\rangle \\
&\equiv\; \langle N,\Psi_1|\,\hat{O}^{(2)}_{\text{tot}}\,|N,\Psi_2\rangle\,.
\end{aligned} \tag{S.6}$$

Now we need to extend this result to generic one-body operators. Any operator $\hat{R}_1$ in the one-particle Hilbert space can be decomposed as $\hat{R}_1 = \sum_{\alpha,\beta}|\alpha\rangle\,R_{\alpha,\beta}\,\langle\beta|$ where $R_{\alpha,\beta}$ are the matrix elements $\langle\alpha|\,\hat{R}_1\,|\beta\rangle$. The definition (4) of first-quantized $\hat{R}^{(1)}_{\rm tot}$ of N particles is obviously linear with respect to the $\hat{R}_1$, thus

$$\hat{R}^{(1)}_{\rm tot} \;=\; \sum_{\alpha,\beta} R_{\alpha,\beta} \sum_{i=1}^{N} \Big(|\alpha\rangle\,\langle\beta| \Big)_{i^{\,\underline{\rm th}}\ \rm particle} \tag{S.7}$$

and therefore, thanks to eq. (S.6),

$$\langle N,\Psi_1|\,\hat{R}^{(1)}_{\rm tot}\,|N,\Psi_2\rangle \;=\; \sum_{\alpha,\beta} R_{\alpha,\beta}\,\langle N,\Psi_1|\,\hat{a}^\dagger_\alpha \hat{a}_\beta\,|N,\Psi_2\rangle \;\; \langle N,\Psi_1|\,\hat{R}^{(2)}_{\rm tot}\,|N,\Psi_2\rangle\,. \qquad \mathcal{Q.E.D.}$$

Problem **1**(c):

Again, we start with a particularly simple 2-body operator $\hat{O}_2 = (|\alpha\rangle\otimes|\beta\rangle)(\langle\gamma|\otimes\langle\delta|)$ which acts on two-particle wave functions according to

$$\begin{aligned}\langle 2,\Psi_1|\,\hat{O}_2\,|2,\Psi_2\rangle \;=& \int d^3\mathbf{x}_1 \int d^3\mathbf{x}_2\; \Psi^*_1(\mathbf{x}_1,\mathbf{x}_2)\phi_\alpha(\mathbf{x}_1)\phi_\beta(\mathbf{x}_2)\\ &\times \int d^3\mathbf{y}_1 \int d^3\mathbf{y}_2\; \phi^*_\gamma(\mathbf{x}_1)\phi^*_\delta(\mathbf{x}_2)\Psi_2(\mathbf{x}_1,\mathbf{x}_2).\end{aligned} \tag{S.8}$$

Consequently, the first-quantized $\hat{O}^{(1)}_{\rm tot}$ operator constructed according to eq. (7) acts in the N-boson Hilbert space as

$$\begin{aligned}\langle N,\Psi_1|\,\hat{O}^{(1)}_{\rm tot}\,|N,\Psi_2\rangle \;=&\\ =&\; \tfrac{1}{2}\sum_{i\neq j} \int d^3\mathbf{x}_1 \cdots \int d^3\mathbf{x}_N\; \Psi^*_1(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_j\ldots,\mathbf{x}_N)\phi_\alpha(\mathbf{x}_i)\phi_\beta(\mathbf{x}_j)\,\times\\ &\times \int d^3\mathbf{x}'_i \int d^3\mathbf{x}'_j\; \phi^*_\gamma(\mathbf{x}'_i)\phi^*_\delta(\mathbf{x}'_j)\Psi_2(\mathbf{x}_1,\ldots,\mathbf{x}'_i,\ldots,\mathbf{x}'_j,\ldots,\mathbf{x}_N)\\ =&\; \frac{N(N-1)}{2} \int d^3\mathbf{x}_1 \cdots \int d^3\mathbf{x}_N\; \Psi^*_1(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2},\mathbf{x}_{N-1},\mathbf{x}_N)\phi_\alpha(\mathbf{x}_{N-1})\phi_\beta(\mathbf{x}_N)\,\times\\ &\times \int d^3\mathbf{x}'_{N-1} \int d^3\mathbf{x}'_N\; \phi^*_\gamma(\mathbf{x}_{N-1})\phi^*_\delta(\mathbf{x}_N)\Psi_2(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2},\mathbf{x}'_{N-1},\mathbf{x}'_N)\\ =&\; \tfrac{1}{2}\int d^3\mathbf{x}_1 \cdots \int d^3\mathbf{x}_{N-2}\; \Psi''^*_1(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2})\;\Psi''_2(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2})\end{aligned} \tag{S.9}$$

where the $(N-2)$–particle wave functions on the last line are

$$\Psi_1''^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2}) \;=\; \sqrt{N(N-1)}\int d^3\mathbf{x}_{N-1}\int d^3\mathbf{x}_N\;\Psi_1^*(\mathbf{x}_1,\ldots,\mathbf{x}_N)\phi_\alpha(\mathbf{x}_{N-1})\phi_\beta(\mathbf{x}_N) \quad \text{(S.10)}$$

and

$$\Psi_2''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2}) \;=\; \sqrt{N(N-1)}\int d^3\mathbf{x}_{N-1}\int d^3\mathbf{x}_N\;\phi_\gamma^*(\mathbf{x}_{N-1})\phi_\delta^*(\mathbf{x}_N)\Psi_2(\mathbf{x}_1,\ldots,\mathbf{x}_N). \quad \text{(S.11)}$$

Notice that the double integral in eq. (S.11) is precisely the integral of eq. (3) applied twice, thus in Fock-space notations

$$\left|(N-2),\Psi_2''\right\rangle \;=\; \hat{a}_\gamma\hat{a}_\delta\left|N,\Psi_2\right\rangle. \quad \text{(S.12)}$$

As to eq. (S.10), it looks like the complex conjugate of eq. (S.11), hence in Fock-space notations

$$\left\langle(N-2),\Psi_1''\right| \;=\; \left\langle N,\Psi_1\right|\hat{a}_\alpha^\dagger\hat{a}_\beta^\dagger. \quad \text{(S.13)}$$

Putting this all together, we arrive at

$$\left\langle N,\Psi_1\right|\hat{O}_{\rm tot}^{(1)}\left|N,\Psi_2\right\rangle \;=\; \left\langle N,\Psi_1\right|\hat{a}_\alpha^\dagger\hat{a}_\beta^\dagger\hat{a}_\gamma\hat{a}_\delta\left|N,\Psi_2\right\rangle \qquad \text{for } \hat{O}_2=(|\alpha\rangle\otimes|\beta\rangle)(\langle\gamma|\otimes\langle\delta|). \quad \text{(S.14)}$$

To extend this result to a general two-body operator $\hat{S}_2$, we use matrix-element decomposition in the two-distinct-particle Hilbert space:

$$\hat{S}_2 \;=\; \sum_{\alpha,\beta,\gamma,\delta} S_{\alpha,\beta,\gamma,\delta}\,\big(|\alpha\rangle\otimes|\beta\rangle\big)\big(\langle\gamma|\otimes\langle\delta|\big) \quad \text{where } S_{\alpha,\beta,\gamma,\delta} \;=\; \langle\alpha|\otimes\langle\beta|\,\hat{S}_2\,|\gamma\rangle\otimes|\delta\rangle. \quad \text{(S.15)}$$

Consequently, similarly to eq. (S.7), the first-quantized form of $\hat{S}_{\rm tot}$ can be written as

$$\hat{S}_{\rm tot}^{(1)} \;=\; \sum_{\alpha,\beta,\gamma,\delta} S_{\alpha,\beta,\gamma,\delta}\,\tfrac{1}{2}\sum_{i\neq j}\Big(|\alpha\rangle\langle\gamma|\Big)_{i^{\underline{\rm th}}\text{ particle}} times\Big(|\beta\rangle\langle\delta|\Big)_{j^{\underline{\rm th}}\text{ particle}}, \quad \text{(S.16)}$$

and therefore

$$\left\langle N,\Psi_1\right|\hat{S}_{\rm tot}^{(1)}\left|N,\Psi_2\right\rangle \;=\; \sum_{\alpha,\beta,\gamma,\delta} S_{\alpha,\beta,\gamma,\delta}\,\left\langle N,\Psi_1\right|\hat{a}_\alpha^\dagger\hat{a}_\beta^\dagger\hat{a}_\gamma\hat{a}_\delta\left|N,\Psi_2\right\rangle \;\equiv\; \left\langle N,\Psi_1\right|\hat{S}_{\rm tot}^{(2)}\left|N,\Psi_2\right\rangle \;\; \mathcal{Q.E.D.}$$

Problem **1**(e):

First, let us calculate the Fock-space commutator $[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta]$. Using the commutators $[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha] = \delta_{\nu.\alpha} \hat{a}^\dagger_\mu$ and $[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}_\beta] = -\delta_{\mu.\beta} \hat{a}_\nu$ (*cf.* previous homework) and applying the Leibniz rule, we have

$$\begin{aligned}
[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta] = {} & [\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha] \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta + \hat{a}^\dagger_\alpha [\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\beta] \hat{a}_\gamma \hat{a}_\delta + \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta [\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}_\gamma] \hat{a}_\delta + \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma [\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}_\delta] \\
= {} & \delta_{\nu\alpha} \hat{a}^\dagger_\mu \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta + \delta_{\nu\beta} \hat{a}^\dagger_\alpha \hat{a}^\dagger_\mu \hat{a}_\gamma \hat{a}_\delta - \delta_{\mu\gamma} \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\nu \hat{a}_\delta - \delta_{\mu\delta} \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\nu \,.
\end{aligned} \tag{S.17}$$

Next, given a second-quantized one-body operator $\hat{A} = \sum_{\mu,\nu} \langle\mu| \hat{A}_1 |\nu\rangle \, \hat{a}^\dagger_\mu \hat{a}_\nu$ and a second-quantized two-body operator $\hat{B} = \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \langle\alpha| \otimes \langle\beta| \hat{B}_2 |\gamma\rangle \otimes |\delta\rangle \, \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta$, we calculate

$$\begin{aligned}
[\hat{A}, \hat{B}] = {} & \sum_{\mu,\nu,\alpha,\beta,\gamma,\delta} \langle\mu| \hat{A}_1 |\nu\rangle \; \langle\alpha\otimes\beta| \hat{B}_2 |\gamma\otimes\delta\rangle \; [\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta] \\
& \langle\!\langle \text{using (S.17)} \rangle\!\rangle \\
= {} & \sum_{\mu,\beta,\gamma,\delta} \hat{a}^\dagger_\mu \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta \times \sum_\nu \langle\mu| \hat{A}_1 |\nu\rangle \langle\nu\otimes\beta| \hat{B}_2 |\gamma\otimes\delta\rangle \\
& + \sum_{\alpha,\mu,\gamma,\delta} \hat{a}^\dagger_\alpha \hat{a}^\dagger_\mu \hat{a}_\gamma \hat{a}_\delta \times \sum_\nu \langle\mu| \hat{A}_1 |\nu\rangle \langle\alpha\otimes\nu| \hat{B}_2 |\gamma\otimes\delta\rangle \\
& - \sum_{\alpha,\beta,\nu,\delta} \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\nu \hat{a}_\delta \times \sum_\mu \langle\alpha\otimes\beta| \hat{B}_2 |\mu\otimes\delta\rangle \langle\mu| \hat{A}_1 |\nu\rangle \\
& - \sum_{\alpha,\beta,\gamma,\nu} \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\nu \times \sum_\mu \langle\alpha\otimes\beta| \hat{B}_2 |\gamma\otimes\mu\rangle \langle\mu| \hat{A}_1 |\nu\rangle \\
& \langle\!\langle \text{renaming summation indices} \rangle\!\rangle \\
= {} & \sum_{\alpha,\beta,\gamma,\delta} \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta \times C_{\alpha,\beta,\gamma,\delta} \,,
\end{aligned} \tag{S.18}$$

where

$$\begin{aligned}
C_{\alpha,\beta,\gamma,\delta} = {} & \sum_\lambda \langle\alpha| \hat{A}_1 |\lambda\rangle \langle\lambda\otimes\beta| \hat{B}_2 |\gamma\otimes\delta\rangle \; + \; \sum_\lambda \langle\beta| \hat{A}_1 |\lambda\rangle \langle\alpha\otimes\lambda| \hat{B}_2 |\gamma\otimes\delta\rangle \\
& - \sum_\lambda \langle\alpha\otimes\beta| \hat{B}_2 |\lambda\otimes\delta\rangle \; \langle\lambda| \hat{A}_1 |\gamma\rangle \; - \; \sum_\lambda \langle\alpha\otimes\beta| \hat{B}_2 |\gamma\otimes\lambda\rangle \; \langle\lambda| \hat{A}_1 |\delta\rangle \\
= {} & \langle\alpha\otimes\beta| \Big(\hat{A}_1(1^{\underline{\mathrm{st}}}) \hat{B}_2 \; + \; \hat{A}_1(2^{\underline{\mathrm{nd}}}) \hat{B}_2 \; - \; \hat{B}_2 \hat{A}_1(1^{\underline{\mathrm{st}}}) \; - \; \hat{B}_2 \hat{A}_1(2^{\underline{\mathrm{nd}}}) \Big) |\gamma\otimes\delta\rangle \\
= {} & \langle\alpha\otimes\beta| \Big[\Big(\hat{A}_1(2^{\underline{\mathrm{nd}}}) + \hat{A}_1(2^{\underline{\mathrm{nd}}}) \Big) , \hat{B}_2 \Big] |\gamma\otimes\delta\rangle \; \equiv \; \langle\alpha\otimes\beta| \hat{C}_2 |\gamma\otimes\delta\rangle \,.
\end{aligned} \tag{S.19}$$

Consequently, $[\hat{A}, \hat{B}] = \hat{C}$. $\mathcal{Q.E.D.}$

Problem **2**(a):

Use product-of-exponentials formula

$$\forall \hat{A}, \hat{B}: \quad e^{\hat{A}} e^{\hat{B}} \;=\; \exp\left(\hat{A} + \hat{B} \,+\, \tfrac{1}{2}[\hat{A}, \hat{B}] \,+\, \tfrac{1}{12}[(\hat{A} - \hat{B}), [\hat{A}, \hat{B}]] \,+\, \cdots\right). \tag{S.20}$$

In particular, for $\hat{A} = \xi \hat{a}^\dagger$, $\hat{B} = \xi^* \hat{a}$ and $[\hat{A}, \hat{B}] = \xi\xi^*$ being a c-number,

$$e^{\xi \hat{a}^\dagger} e^{-\xi^* \hat{a}} \;=\; \exp\left(\xi \hat{a}^\dagger - \xi^* \hat{a} + \tfrac{1}{2}\xi\xi^*\right), \quad \text{exactly,} \tag{S.21}$$

and therefore

$$|\xi\rangle \;\stackrel{\text{def}}{=}\; e^{\xi \hat{a}^\dagger - \xi^* \hat{a}}\,|0\rangle \;=\; e^{-|\xi|^2/2} e^{\xi \hat{a}^\dagger} e^{-\xi^* \hat{a}}\,|0\rangle \;=\; e^{-|\xi|^2/2} e^{\xi \hat{a}^\dagger}\,|0\rangle\,. \tag{S.22}$$

(Note $e^{-\xi^* \hat{a}}\,|0\rangle = |0\rangle$ since $\hat{a}\,|0\rangle = 0$.)

Next, $[\hat{a}, \hat{a}^\dagger] = 1$ implies that for any function $f(\hat{a}^\dagger)$, $[\hat{a}, f(\hat{a}^\dagger)] = f'(\hat{a}^\dagger)$. In particular, $[\hat{a}, e^{\xi \hat{a}^\dagger}] = \xi e^{\xi \hat{a}^\dagger}$ or in other words, $(\hat{a} - \xi) e^{\xi \hat{a}^\dagger} = e^{\xi \hat{a}^\dagger} \hat{a}$ and hence $(\hat{a} - \xi)\,|\xi\rangle \propto e^{\xi \hat{a}^\dagger} \hat{a}\,|0\rangle = 0$. $\mathcal{Q.E.D.}$

Problem **2**(b):

For any *normal-ordered* product of creation and annihilation operators — *i.e.*, a product in which all creation operators are to the right of all annihilation operators — one has $\langle\xi|\,(\hat{a}^\dagger)^k (\hat{a})^\ell\,|\xi\rangle = (\xi^*)^k \xi^\ell$, simply because $\hat{a}\,|\xi\rangle = \xi\,|\xi\rangle$ and $\langle\xi|\,\hat{a}^\dagger = \xi^*\,\langle\xi|$. In particular, $\langle\xi|\,(\hat{n} = \hat{a}^\dagger \hat{a})\,|\xi\rangle = \xi^* \xi$. On the other hand,

$$\hat{n}^2 \;=\; \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \;=\; \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \;+\; \hat{a}^\dagger \hat{a} \;\Longrightarrow\; \langle\xi|\,\hat{n}^2\,|\xi\rangle \;=\; (\xi^*)^2 \xi^2 \;+\; \xi^* \xi \;=\; \bar{n}^2 \;+\; \bar{n} \tag{S.23}$$

hence $\Delta n = \sqrt{\langle \hat{n}^2 \rangle - \bar{n}^2} = \sqrt{\bar{n}}$.

In a similar manner,

$$\hat{q} \;=\; \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger), \quad \hat{q}^2 \;=\; \frac{\hbar}{2m\omega}\left((\hat{a})^2 + (\hat{a}^\dagger)^2 + 2\hat{a}^\dagger \hat{a} + 1\right) \;\Longrightarrow$$

$$\langle\xi|\,\hat{q}^2\,|\xi\rangle \;=\; \frac{\hbar}{2m\omega}\big((\xi+\xi^*)^2+1\big) \;=\; \langle\xi|\,\hat{q}\,|\xi\rangle^2 \;+\; \frac{\hbar}{2m\omega}$$

and likewise

$$\langle\xi|\,\hat{p}^2\,|\xi\rangle \;=\; \frac{m\omega\hbar}{2}\big((-i\xi+i\xi^*)^2+1\big) \;=\; \langle\xi|\,\hat{p}\,|\xi\rangle^2 \;+\; \frac{m\omega\hbar}{2}\,,$$

thus

$$\Delta q \;=\; \sqrt{\frac{\hbar}{2m\omega}}\,, \quad \Delta p \;=\; \sqrt{\frac{m\omega\hbar}{2}}\,, \quad \Delta q\Delta p \;=\; \frac{\hbar}{2}\,. \tag{S.24}$$

$\mathcal{Q.E.D.}$

Problem **2**(c):

In the Schrödinger picture, $\hat{a}^\dagger$ is time independent, hence $(d/dt)e^{\xi\hat{a}^\dagger} = (d\xi/dt)\hat{a}^\dagger e^{\xi\hat{a}^\dagger}$. Using time independence of the magnitude $|\xi|$, we then have

$$\frac{d}{dt}\,\Big(|\xi\rangle = e^{-|\xi|^2/2}e^{\xi\hat{a}^\dagger}\,|0\rangle\Big) \;=\; \frac{d\xi}{dt}\,\hat{a}^\dagger\,|\xi\rangle \;=\; \frac{1}{\xi}\frac{d\xi}{dt}\,\hat{a}^\dagger\hat{a}\,|\xi\rangle \;=\; -i\omega\hat{a}^\dagger\hat{a}\,|\xi\rangle \tag{S.25}$$

where the last equality comes from $\xi(t) = \xi_0 e^{-i\omega t}$. In other words,

$$i\hbar\frac{d}{dt}\,|\xi(t)\rangle \;=\; \hbar\omega\hat{a}^\dagger\hat{a}\,|\xi(t)\rangle \;\equiv\; \hat{H}\,|\xi(t)\rangle\,. \tag{S.26}$$

$\mathcal{Q.E.D.}$

Problem **2**(d):

In question **2**(a) we saw that $[\hat{a},\hat{a}^\dagger]=1$ implies $e^{\xi\hat{a}^\dagger}\hat{a} = (\hat{a}-\xi)e^{\xi\hat{a}^\dagger}$ for any c-number ξ. Iterating this identity gives us $e^{\xi\hat{a}^\dagger}f(\hat{a}) = f(\hat{a}-\xi)e^{\xi\hat{a}^\dagger}$ for any function $f(\hat{a})$ of the annihilation operator, and in particular

$$e^{\xi\hat{a}^\dagger}\,e^{\eta^*\hat{a}} \;=\; e^{\eta^*(\hat{a}-\xi)}\,e^{\xi\hat{a}^\dagger} \;=\; e^{-\eta^*\xi}\,e^{\eta^*\hat{a}}\,e^{\xi\hat{a}^\dagger}. \tag{S.27}$$

Consequently, the quantum overlap of the coherent states $|\xi\rangle$ and $\langle\eta|$ is

$$\begin{aligned}
\langle\eta|xi\rangle \;&=\; e^{-|\eta|^2/2}e^{-|\xi|^2/2}\,\langle 0|\,e^{\eta^*\hat{a}}e^{\xi\hat{a}^\dagger}\,|0\rangle \\
&=\; e^{-|\eta|^2/2}e^{-|\xi|^2/2}e^{-\eta^*\xi}\,\langle 0|\,e^{\xi\hat{a}^\dagger}\,e^{\eta^*\hat{a}}\,|0\rangle \\
&=\; \exp\big(-\tfrac{1}{2}|\eta|^2 - \tfrac{1}{2}|\xi|^2 - \eta^*\xi\big)
\end{aligned} \tag{S.28}$$

because $e^{\eta^*\hat{a}}\,|0\rangle = |0\rangle$, $\langle 0|\,e^{\xi\hat{a}^\dagger} = \langle 0|$ and $\langle 0|0\rangle = 1$. In terms of the probability overlap,

$$|\langle\eta|\xi\rangle|^2 \;=\; e^{-|\eta-\xi|^2}. \tag{S.29}$$

Problem **2**(e):

Generalization of coherent states to multi-oscillatory systems and further to the creation / annihilation fields is completely straightforward:

$$|\text{coherent}\rangle \;\stackrel{\text{def}}{=}\; \exp\bigl(\hat{F}^\dagger - \hat{F}\bigr)\,|0\rangle \;=\; e^{-\bar{N}/2}\,e^{\hat{F}^\dagger}\,|0\rangle \tag{S.30}$$

where

$$\hat{F}^\dagger \;=\; \xi\hat{a}^\dagger \;\to\; \sum_\alpha \xi_\alpha \hat{a}^\dagger_\alpha \;\to\; \int\! d^3\mathbf{x}\,\Phi(\mathbf{x})\hat{\Psi}^\dagger(\mathbf{x}). \tag{S.31}$$

Similar to the single-oscillator theory, $\bigl(\hat{\Psi}(\mathbf{x}) - \Phi(\mathbf{x})\bigr)e^{\hat{F}^\dagger} = e^{\hat{F}^\dagger}\hat{\Psi}^\dagger(\mathbf{x})$, hence

$$\hat{\Psi}(\mathbf{x})\,|\Phi\rangle \;=\; \Phi(\mathbf{x})\,|\Phi\rangle\,. \tag{S.32}$$

Problem **2**(f):

Using eq. (S.32) and its hermitian conjugate, we have

$$\langle\Phi|\,\hat{\Psi}^\dagger(\mathbf{x}1)\cdots\hat{\Psi}^\dagger(\mathbf{x}_k)\hat{\Psi}(\mathbf{y}_1)\cdots\hat{\Psi}(\mathbf{y}_\ell)\,|\Phi\rangle \;=\; \Phi^*(\mathbf{x}_1)\cdots\Phi^*(\mathbf{x}_k)\Phi(\mathbf{y}_1)\cdots\Phi(\mathbf{y}_\ell) \tag{S.33}$$

for any *normal-ordered* product of the quantum fields. Specifically, for the particle-number operator $\hat{N}$ we have eq. (12), while for its square — whose normal-ordered form

$$\hat{N}^2 \;=\; \iint\! d^3\mathbf{x}\,d^3\mathbf{y}\;\hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}^\dagger(\mathbf{y})\hat{\Psi}(\mathbf{x})\hat{\Psi}(\mathbf{y}) \;+\; \int\! d^3\mathbf{x}\;\hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}(\mathbf{x}) \tag{S.34}$$

generalizes eq. (S.23) — we have

$$\langle\Phi|\,\hat{N}^2\,|\Phi\rangle \;=\; \iint\! d^3\mathbf{x}\,d^3\mathbf{y}\;\Phi^*(\mathbf{x})\Phi^*(\mathbf{y})\Phi(\mathbf{x})\Phi(\mathbf{y}) \;+\; \int\! d^3\mathbf{x}\;\Phi^*(\mathbf{x})\Phi(\mathbf{x}) \;=\; \langle\Phi|\,\hat{N}\,|\Phi\rangle^2 \;+\; \langle\Phi|\,\hat{N}\,|\Phi\rangle\,, \tag{S.35}$$

and hence $\Delta N = \sqrt{\bar{N}}$, $\quad\mathcal{Q.E.D.}$

Problem **2**(g):

First of all, if $\Phi(\mathbf{x},t)$ satisfies the classical field equation — which looks exactly like a one-particle Schrödinger equation — then $\bar{N}$ remains constant. (This is undergraduate-level QM.) Also, in the Schrödinger picture of the QFT,

$$\frac{d}{dt}\, e^{\hat{F}^\dagger} \;=\; \frac{d\hat{F}^\dagger}{dt}\, e^{\hat{F}^\dagger} \;=\; \left[\int\! d^3\mathbf{x}\, \frac{\partial \Phi(\mathbf{x},t)}{\partial t}\, \hat{\Psi}^\dagger(\mathbf{x})\right] e^{\hat{F}^\dagger} \tag{S.36}$$

thanks to mutual commutativity of the creation fields. Consequently, exactly as in question (c),

$$\begin{aligned}
i\hbar\frac{d}{dt}\left(|\Phi\rangle = e^{-\bar{N}/2} e^{\hat{F}^\dagger}\,|0\rangle\right) \;&=\; \left[\int\! d^3\mathbf{x}\, i\hbar\frac{\partial \Phi(\mathbf{x},t)}{\partial t}\, \hat{\Psi}^\dagger(\mathbf{x})\right] |\Phi\rangle \\
&\langle\!\langle\text{using the classical field equation for } \Phi\rangle\!\rangle \\
&=\; \left[\int\! d^3\mathbf{x}\, \left(\frac{-\hbar^2}{2M}\nabla^2 + V(\mathbf{x})\right) \Phi(\mathbf{x})\, \hat{\Psi}^\dagger(\mathbf{x})\right] |\Phi\rangle \\
&\langle\!\langle\text{using coherence}\rangle\!\rangle \\
&=\; \left[\int\! d^3\mathbf{x}\, \hat{\Psi}^\dagger(\mathbf{x}) \left(\frac{-\hbar^2}{2M}\nabla^2 + V(\mathbf{x})\right) \hat{\Psi}(\mathbf{x})\right] |\Phi\rangle \\
&=\; \hat{H}\,|\Phi\rangle\,.
\end{aligned} \tag{S.37}$$

$\mathcal{Q.E.D.}$

Problem **2**(h):

Generalizing (d) to multi-oscillatory systems is completely straightforward:

$$|\langle\eta|\xi\rangle|^2 \;=\; \prod_\alpha e^{-|\xi_\alpha-\eta_\alpha|^2} \;=\; \exp\left(-\sum_\alpha |\xi_\alpha-\eta_\alpha|^2\right)$$

or for the field theory,

$$|\langle\Phi_1|\Phi_2\rangle|^2 \;=\; \exp\left(\int\! d^3\mathbf{x}\; |\Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})|^2\right), \tag{S.38}$$

which is exponentially small for any macroscopic $\delta\Phi(\mathbf{x}) = \Phi_1(x) - \Phi_2(x)$. Indeed, a *macroscopic* difference between two coherent states means (by definition) that $\delta\Phi$ affects a large number of particles, $\int|\delta\Phi|^2 \gg 1$ and hence an *exponentially* tiny overlap (S.38).