Problem **1**(a):

The linear sigma model has scalar potential

$$V(\sigma,\underset{\sim}{\pi}) \;=\; \frac{\lambda}{8}\left(\sigma^2+\underset{\sim}{\pi}^2-f^2\right)^2 \;-\; \beta\sigma. \tag{S.1}$$

Any local minimum of this potential satisfies

$$\begin{aligned}
\frac{\partial V}{\partial\underset{\sim}{\pi}} \;&=\; \frac{\lambda}{2}\left(\sigma^2+\underset{\sim}{\pi}^2-f^2\right)\times\underset{\sim}{\pi} \;=\; 0,\text{and}\\
\frac{\partial V}{\partial\sigma} \;&=\; \frac{\lambda}{2}\left(\sigma^2+\underset{\sim}{\pi}^2-f^2\right)\times\sigma \;-\; \beta \;=\; 0,
\end{aligned} \tag{S.2}$$

which together imply $\underset{\sim}{\pi}=0$ while σ satisfies a cubic equation

$$\sigma(\sigma^2-f^2) \;-\; \frac{2\beta}{\lambda} \;=\; 0. \tag{S.3}$$

For small $\beta>0$ this equation has three real solutions:

$$\begin{aligned}
\sigma_1 \;&\approx\; -\frac{2\beta}{\lambda f^2} && \text{which is a local maximum,}\\
\sigma_2 \;&\approx\; -f \;+\; \frac{\beta}{\lambda f^2} && \text{which is a saddle point,}\\
\sigma_3 \;&\approx\; +f \;+\; \frac{\beta}{\lambda f^2} && \text{which is the only minimum.}
\end{aligned} \tag{S.4}$$

Problem **1**(b):

Let us shift the σ field by its vacuum expectation value $\sigma(x)=\langle\sigma\rangle+\tilde{\sigma}(x)$. Rewriting the potential in terms of the $\tilde{\sigma}$ and $\underset{\sim}{\pi}$ fields, we have

$$V \;=\; \text{const} \;+\; \frac{\beta}{2\langle\sigma\rangle}(\tilde{\sigma}^2+\underset{\sim}{\pi}^2) \;+\; \frac{\lambda\langle\sigma\rangle^2}{2}\,\tilde{\sigma}^2 \;+\; \frac{\lambda\langle\sigma\rangle}{2}\,\tilde{\sigma}(\tilde{\sigma}^2+\underset{\sim}{\pi}^2) \;+\; \frac{\lambda}{8}(\tilde{\sigma}^2+\underset{\sim}{\pi}^2)^2. \tag{S.5}$$

Disregarding the constant part, we split the Lagrangian (1) as

$$\mathcal{L} \;=\; \widetilde{\mathcal{L}}_{\text{free}} \;+\; \widetilde{\mathcal{L}}_{\text{int}} \tag{S.6}$$

where the quadratic part

$$\widetilde{\mathcal{L}}_{\text{free}} \;=\; \tfrac{1}{2}(\partial_\mu\tilde{\sigma})^2 \;+\; \tfrac{1}{2}(\partial\mu\underset{\sim}{\pi})^2 \;-\; \frac{\beta}{2\,\langle\sigma\rangle}(\tilde{\sigma}^2+\underset{\sim}{\pi}^2) \;-\; \frac{\lambda\,\langle\sigma\rangle^2}{2}\,\tilde{\sigma}^2 \tag{S.7}$$

corresponds to the free $\tilde{\sigma}$ and $\underset{\sim}{\pi}$ fields while the cubic and quartic terms

$$\widetilde{\mathcal{L}}_{\text{int}} \;=\; -\frac{\lambda\,\langle\sigma\rangle}{2}\,\tilde{\sigma}(\tilde{\sigma}^2+\underset{\sim}{\pi}^2) \;-\; \frac{\lambda}{8}(\tilde{\sigma}^2+\underset{\sim}{\pi}^2)^2. \tag{S.8}$$

govern their interactions. Particles' masses follow from the quadratic Lagrangian (S.7):

$$\begin{aligned} M_\pi^2 \;&=\; \frac{\beta}{\langle\sigma\rangle} \;\approx\; \frac{\beta}{f}, \quad \text{and} \\ M_\sigma^2 \;&=\; \frac{\beta}{\langle\sigma\rangle} \;+\; \lambda\,\langle\sigma\rangle^2 \;\approx\; \lambda f^2 \;\gg\; M_\pi^2\,. \end{aligned} \qquad \mathcal{Q.E.D.}$$

Problem **2**(a):

For the sake notational simplicity, let us use discrete particle momenta in this problem, hence discrete bosonic commutations relations

$$[\hat{a}_{\mathbf{p}},\hat{a}_{\mathbf{p}'}] \;=\; 0, \quad [\hat{a}^\dagger_{\mathbf{p}},\hat{a}^\dagger_{\mathbf{p}'}] \;=\; 0, \quad [\hat{a}_{\mathbf{p}},\hat{a}^\dagger_{\mathbf{p}'}] \;=\; \delta_{\mathbf{p},\mathbf{p}'}\,. \tag{S.9}$$

Taking the Bogolyubov transform formulae (3) as definitions of the $\hat{b}_{\mathbf{p}}$ and $\hat{b}^\dagger_{\mathbf{p}}$ operators, we calculate

$$[\hat{b}_{\mathbf{p}},\hat{b}_{\mathbf{p}'}] \;=\; \cosh(t_{\mathbf{p}})\sinh(t_{\mathbf{p}'})\delta_{\mathbf{p},-\mathbf{p}'} \;-\; \sinh(t_{\mathbf{p}})\cosh(t_{\mathbf{p}'})\delta_{-\mathbf{p},\mathbf{p}'} \;=\; 0 \tag{S.10}$$

because $t_{\mathbf{p}} = t_{\mathbf{p}'}$ for $\mathbf{p} = -\mathbf{p}'$. Likewise, $[\hat{b}^\dagger_{\mathbf{p}},\hat{b}^\dagger_{\mathbf{p}'}] = 0$. Finally,

$$\begin{aligned} [\hat{b}_{\mathbf{p}},\hat{b}_{\mathbf{p}'}] \;&=\; \cosh(t_{\mathbf{p}})\cosh(t_{\mathbf{p}'})\delta_{\mathbf{p},\mathbf{p}'} \;-\; \sinh(t_{-\mathbf{p}})\sinh(t_{-\mathbf{p}'})\delta_{-\mathbf{p},-\mathbf{p}'} \\ &=\; \delta_{\mathbf{p}.\mathbf{p}'}\Big(\cosh^2(t_{\mathbf{p}}) - \sinh^2(t_{\mathbf{p}}) = 1\Big). \end{aligned} \tag{S.11}$$

In other words, the $\hat{b}_{\mathbf{p}}$ and $\hat{b}^\dagger_{\mathbf{p}}$ operators satisfy the same bosonic commutations relations

$$[\hat{b}_{\mathbf{p}}, \hat{b}_{\mathbf{p}'}] \;=\; 0, \quad [\hat{b}^\dagger_{\mathbf{p}}, \hat{b}^\dagger_{\mathbf{p}'}] \;=\; 0, \quad [\hat{b}_{\mathbf{p}}, \hat{b}^\dagger_{\mathbf{p}'}] \;=\; \delta_{\mathbf{p},\mathbf{p}'}\,. \tag{S.12}$$

as the original $\hat{a}_{\mathbf{p}}$ and $\hat{a}^\dagger_{\mathbf{p}}$ operators. $\mathcal{Q.E.D.}$

Problem **2**(b):

A straightforward calculation shows that

$$\sum_{\mathbf{p}} \omega_{\mathbf{p}} \hat{b}^\dagger_{\mathbf{p}} \hat{b}_{\mathbf{p}} \;= \sum_{\mathbf{p}} \omega_{\mathbf{p}} \cosh(2t_{\mathbf{p}}) \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} \;+\; \tfrac{1}{2} \sum_{\mathbf{p}} \omega_{\mathbf{p}} \sinh(2t_{\mathbf{p}}) \left(\hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{-\mathbf{p}} \;+\; \hat{a}_{-\mathbf{p}} \hat{a}_{\mathbf{p}} \right) \;+\; \text{const.} \tag{S.13}$$

Therefore, the Hamiltonian (4) can be "diagonalized" in terms of the transformed creation / annihilation operators (3) if and only if we can solve for $\omega_{\mathbf{p}}$ and $t_{\mathbf{p}}$ such that

$$\omega_{\mathbf{p}} \cosh(2t_{\mathbf{p}}) \;=\; A_{\mathbf{p}} \quad \text{and} \quad \omega_{\mathbf{p}} \sinh(2t_{\mathbf{p}}) \;=\; B_{\mathbf{p}}\,. \tag{S.14}$$

The latter equations are solvable whenever $A_{\mathbf{p}} > |B_{\mathbf{p}}|$ and the solution is

$$t_{\mathbf{p}} \;=\; \frac{1}{2} \operatorname{artanh} \frac{B_{\mathbf{p}}}{A_{\mathbf{p}}} \quad \text{and} \quad \omega_{\mathbf{p}} \;=\; \sqrt{A^2_{\mathbf{p}} - B^2_{\mathbf{p}}}\,. \qquad \mathcal{Q.E.D.}$$

Problem **3**(a):

In the Hamiltonian formalism for the classical fields $\Phi(x)$ and $\Phi^*(x)$, the canonical conjugate fields are

$$\Pi(x) \;=\; \frac{\partial \mathcal{L}}{\partial(\partial_0 \Phi^*)} \;=\; \partial_0 \Phi(x) \qquad \text{and} \quad \Pi^*(x) \;=\; \frac{\partial \mathcal{L}}{\partial(\partial_0 \Phi)} \;=\; \partial_0 \Phi^*(x). \tag{S.15}$$

The canonical conjugation implies canonical Poisson brackets between the classical fields $\Phi(\mathbf{x})$ and $\Pi^*(\mathbf{x})$, and likewise $\Phi^*(\mathbf{x})$ and $\Pi(\mathbf{x})$ and hence the canonical commutation relation between their quantum counterparts: In the Schrödinger picture

$$[\hat{\Phi}(\mathbf{x}), \hat{\Phi}(\mathbf{x}')] \;=\; [\hat{\Phi}(\mathbf{x}), \hat{\Phi}^\dagger(\mathbf{x}')] \;=\; [\hat{\Phi}^\dagger(\mathbf{x}), \hat{\Phi}^\dagger(\mathbf{x}')] \;=\; 0,$$

$$\begin{aligned} [\hat\Pi(\mathbf{x}),\hat\Pi(\mathbf{x}')] \;&=\; [\hat\Pi(\mathbf{x}),\hat\Pi^\dagger(\mathbf{x}')] \;=\; [\hat\Pi^\dagger(\mathbf{x}),\hat\Pi^\dagger(\mathbf{x}')] \;=\; 0,\\ [\hat\Phi(\mathbf{x}),\hat\Pi(\mathbf{x}')] \;&=\; [\hat\Phi^\dagger(\mathbf{x}),\hat\Pi^\dagger(\mathbf{x}')] \;=\; 0,\\ [\hat\Phi(\mathbf{x}),\hat\Pi^\dagger(\mathbf{x}')] \;&=\; [\hat\Phi^\dagger(\mathbf{x}),\hat\Pi(\mathbf{x}')] \;=\; i\delta^{(3)}(\mathbf{x}-\mathbf{x}'). \end{aligned} \tag{S.16}$$

In the Heisenberg picture, we have similar commutation relations for equal times $t = t'$; for un-equal times, the formulae are much more complicated, *cf.* problem 4.

Classically, the Hamiltonian density is

$$\begin{aligned} \mathcal{H} \;&=\; \Pi\partial_0\Phi \;+\; \Pi^*\partial_0\Phi^* \;-\; \mathcal{L}\\ &=\; \Pi^*\Pi \;+\; \nabla\Phi^*\cdot\nabla\Phi \;+\; m^2\,\Phi^*\Phi, \end{aligned} \tag{S.17}$$

so the quantum theory's Hamiltonian is obviously (9) (modulo operator-ordering ambiguity). $\mathcal{Q.E.D}$.

Problem **3**(b):

Fourier transforming the canonical commutation relations (S.16) results in

$$\begin{aligned} [\hat\Phi_{\mathbf{p}},\hat\Phi_{\mathbf{p}'}] \;&=\; [\hat\Phi_{\mathbf{p}},\hat\Phi^\dagger_{\mathbf{p}'}] \;=\; [\hat\Phi^\dagger_{\mathbf{p}},\hat\Phi^\dagger_{\mathbf{p}'}] \;=\; 0,\\ [\hat\Pi_{\mathbf{p}},\hat\Pi_{\mathbf{p}'}] \;&=\; [\hat\Pi_{\mathbf{p}},\hat\Pi^\dagger_{\mathbf{p}'}] \;=\; [\hat\Pi^\dagger_{\mathbf{p}},\hat\Pi^\dagger_{\mathbf{p}'}] \;=\; 0,\\ [\hat\Phi_{\mathbf{p}},\hat\Pi_{\mathbf{p}'}] \;&=\; [\hat\Phi^\dagger_{\mathbf{p}},\hat\Pi^\dagger_{\mathbf{p}'}] \;=\; 0,\\ [\hat\Phi_{\mathbf{p}},\hat\Pi^\dagger_{\mathbf{p}'}] \;&=\; [\hat\Phi^\dagger_{\mathbf{p}},\hat\Pi_{\mathbf{p}'}] \;=\; (2\pi)^3 i\delta^{(3)}(\mathbf{p}-\mathbf{p}'). \end{aligned} \tag{S.18}$$

Consequently,

$$[\hat a_{\mathbf{p}},\hat a_{\mathbf{p}'}] \;=\; [\hat a_{\mathbf{p}},\hat b^\dagger_{\mathbf{p}'}] \;=\; [\hat b^\dagger_{\mathbf{p}},\hat b^\dagger_{\mathbf{p}'}] \;=\; 0 \tag{S.19}$$

because all $\Phi_{\mathbf{p}}$ and all $\Pi_{\mathbf{p}}$ commute with each other. Similarly,

$$[\hat b_{\mathbf{p}},\hat b_{\mathbf{p}'}] \;=\; [\hat b_{\mathbf{p}},\hat a^\dagger_{\mathbf{p}'}] \;=\; [\hat a^\dagger_{\mathbf{p}},\hat a^\dagger_{\mathbf{p}'}] \;=\; 0 \tag{S.20}$$

because all $\Phi^\dagger_{\mathbf{p}}$ and all $\Pi^\dagger_{\mathbf{p}}$ commute with each other too. Less obviously,

$$[\hat a_{\mathbf{p}},\hat b_{\mathbf{p}'}] \;=\; iE_{\mathbf{p}}\times(2\pi)^3 i\delta^{(3)}(\mathbf{p}+\mathbf{p}') \;+\; iE_{\mathbf{p}'}\times(2\pi)^3(-i)\delta^{(3)}(-\mathbf{p}-\mathbf{p}') \;=\; 0, \tag{S.21}$$

and likewise $[\hat{a}^\dagger_{\mathbf{p}}, \hat{b}^\dagger_{\mathbf{p}'}] = 0$. Finally,

$$
\begin{aligned}
[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{p}'}] \;=\; [\hat{b}_{\mathbf{p}}, \hat{b}^\dagger_{\mathbf{p}'}] \;&=\; -iE_{\mathbf{p}} \times (2\pi)^3 i\delta^{(3)}(\mathbf{p}-\mathbf{p}') \;+\; iE_{\mathbf{p}'} \times (2\pi)^3 (-i)\delta^{(3)}(\mathbf{p}'-\mathbf{p}) \\
&=\; 2E_{\mathbf{p}} \times (2\pi)^3\, \delta^{(3)}(\mathbf{p}-\mathbf{p}'),
\end{aligned}
\tag{S.22}
$$

as appropriate for the relativistic normalization of the one-particle states, $\langle \mathbf{p}|\mathbf{p}'\rangle = 2E_{\mathbf{p}} \times (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{p}')$. $\mathcal{Q.E.D.}$

<u>Problem **3**(c)</u>:

First, Fourier-transforming the free Hamiltonian (9) gives us

$$
\hat{H}_{\rm free} \;=\; \int\! \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\hat{\Pi}^\dagger_{\mathbf{p}} \hat{\Pi}_{\mathbf{p}} \;+\; E^2_{\mathbf{p}}\, \hat{\Phi}^\dagger_{\mathbf{p}} \hat{\Phi}_{\mathbf{p}} \right). \tag{S.23}
$$

Second, we reverse the definitions (11) to obtain

$$
\hat{\Phi}_{\mathbf{p}} \;=\; \frac{\hat{a}_{\mathbf{p}} + \hat{b}^\dagger_{-\mathbf{p}}}{2E_{\mathbf{p}}} \qquad \text{and} \quad \hat{\Pi}_{\mathbf{p}} \;=\; \frac{\hat{b}_{\mathbf{p}} - \hat{a}^\dagger_{-\mathbf{p}}}{2i}\,. \tag{S.24}
$$

Third, we calculate

$$
\begin{aligned}
E^2_{\mathbf{p}}\, \hat{\Phi}^\dagger_{\mathbf{p}} \hat{\Phi}_{\mathbf{p}} \;+\; \hat{\Pi}_{-\mathbf{p}} \hat{\Pi}^\dagger_{-\mathbf{p}} \;&=\; \tfrac{1}{4}\big(\hat{a}^\dagger_{\mathbf{p}} + \hat{b}_{-\mathbf{p}}\big)\big(\hat{a}_{\mathbf{p}} + \hat{b}^\dagger_{-\mathbf{p}}\big) \;+\; \tfrac{1}{4}\big(\hat{a}^\dagger_{\mathbf{p}} - \hat{b}_{-\mathbf{p}}\big)\big(\hat{a}_{\mathbf{p}} - \hat{b}^\dagger_{-\mathbf{p}}\big) \\
&=\; \tfrac{1}{2}\Big(\hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} \;+\; \hat{b}_{-\mathbf{p}} \hat{b}^\dagger_{-\mathbf{p}}\Big).
\end{aligned}
\tag{S.25}
$$

Finally, we put eqs. (S.25) and (S.23) together and derive

$$
\begin{aligned}
\hat{H}_{\rm free} \;&=\; \int\! \frac{d^3\mathbf{p}}{(2\pi)^3} \left(E^2_{\mathbf{p}}\, \hat{\Phi}^\dagger_{\mathbf{p}} \hat{\Phi}_{\mathbf{p}} + \; \hat{\Pi}_{-\mathbf{p}} \hat{\Pi}^\dagger_{-\mathbf{p}} \right) \\
&=\; \int\! \frac{d^3\mathbf{p}}{(2\pi)^3}\, \frac{1}{2} \Big(\hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} \;+\; \hat{b}_{-\mathbf{p}} \hat{b}^\dagger_{-\mathbf{p}} \Big) \\
&=\; \int\! \frac{d^3\mathbf{p}}{(2\pi)^{,} 2E_{\mathbf{p}}} \big(E_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} \;+\; E_{\mathbf{p}} \hat{b}^\dagger_{\mathbf{p}} \hat{b}_{\mathbf{p}} \big) \;+\; \text{const.}
\end{aligned}
\tag{S.26}
$$

$\mathcal{Q.E.D.}$

Problem **3**(d):

In the Hamiltonian formalism, the classical charge density is

$$J^0 \;=\; i\Phi^*(\partial^0\Phi) - i(\partial^0\Phi^*)\Phi \;=\; i\Phi^*\Pi \;-\; i\Phi\Pi^*, \tag{S.27}$$

(*cf.* eq. (7)). In a quantum theory, this gives us the charge operator

$$\hat{Q} \;=\int\! d^3\mathbf{x}\,\hat{J}^0(\mathbf{x}) \;=\int\! d^3\mathbf{x}\,\left(\tfrac{i}{2}\{\hat{\Phi}^\dagger,\hat{\Pi}\} \;-\; \tfrac{i}{2}\{\hat{\Phi},\hat{\Pi}^\dagger\} \;+\; \text{const}\right), \tag{S.28}$$

where the unknown constant reflects the ambiguity of operator ordering in quantizing a classical quantity. We shall see momentarily that setting this constant to zero results in the normal ordering of $\hat{Q}$ in terms of creation and annihilation operators and hence in zero charge of the vacuum state $|0\rangle$. Physically, this is what we want — hence, the first eq. (13).

Our next step is to Fourier transforming eq. (S.28), which yields

$$\hat{Q} \;=\int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\left(\tfrac{i}{2}\{\hat{\Phi}^\dagger_{\mathbf{p}},\hat{\Pi}_{\mathbf{p}}\} \;-\; \tfrac{i}{2}\{\hat{\Phi}_{\mathbf{p}},\hat{\Pi}^\dagger_{\mathbf{p}}\}\right). \tag{S.29}$$

Furthermore, simple algebra gives

$$\begin{aligned}
\tfrac{i}{2}\{\hat{\Phi}^\dagger_{\mathbf{p}},\hat{\Pi}_{\mathbf{p}}\} \;&=\; \frac{\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \;-\; \hat{b}^\dagger_{-\mathbf{p}}\hat{b}_{-\mathbf{p}} \;+\; \hat{a}^\dagger_{\mathbf{p}}\hat{b}^\dagger_{-\mathbf{p}} \;-\; \hat{a}_{\mathbf{p}}\hat{b}_{-\mathbf{p}}}{4E_{\mathbf{p}}}\,,\\
\tfrac{-i}{2}\{\hat{\Phi}_{\mathbf{p}},\hat{\Pi}^\dagger_{\mathbf{p}}\} \;&=\; \frac{\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \;-\; \hat{b}^\dagger_{-\mathbf{p}}\hat{b}_{-\mathbf{p}} \;-\; \hat{a}^\dagger_{\mathbf{p}}\hat{b}^\dagger_{-\mathbf{p}} \;+\; \hat{a}_{\mathbf{p}}\hat{b}_{-\mathbf{p}}}{4E_{\mathbf{p}}}\,,
\end{aligned} \tag{S.30}$$

(*cf.* eq. (S.24)) and therefore,

$$\hat{Q} \;=\int \frac{d^3\mathbf{p}}{(2\pi)^3\,2E_{\mathbf{p}}}\,\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \;-\; \hat{b}^\dagger_{-\mathbf{p}}\hat{b}_{-\mathbf{p}}\right) \;=\int \frac{d^3\mathbf{p}}{(2\pi)^3\,2E_{\mathbf{p}}}\,\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \;-\; \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}}\right). \qquad \mathcal{Q.E.D.}$$

Problem **3**(e):

According to eq. (14), classically

$$T^{0i} \;=\; \partial^0\Phi^*\,\partial^i\Phi \,+\, \partial^0\Phi\,\partial^i\Phi^* \;=\; -\Pi^*\,\partial_i\Phi \,-\, \Pi\,\partial_i\Phi^*, \tag{S.31}$$

and hence in the quantum theory

$$\hat{\mathbf{P}}_{\rm mech} \;=\; \int\! d^3\mathbf{x}\,\left(-\tfrac12\{\hat\Pi^\dagger,\nabla\hat\Phi\} - \tfrac12\{\hat\Pi,\nabla\hat\Phi^\dagger\}\right). \tag{S.32}$$

Fourier-transforming this formula, we arrive at

$$\hat{\mathbf{P}}_{\rm mech} \;=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\left(-\frac{i\mathbf{p}}{2}\{\hat\Pi^\dagger_{\mathbf{p}},\hat\Phi_{\mathbf{p}}\} \,+\, \frac{i\mathbf{p}}{2}\{\hat\Pi_{\mathbf{p}},\hat\Pi^\dagger_{\mathbf{p}}\}\right), \tag{S.33}$$

and then applying eqs. (S.30) finally yields

$$\hat{\mathbf{P}}_{\rm mech} \;=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3\,2E_{\mathbf{p}}}\,\left(\mathbf{p}\,\hat a^\dagger_{\mathbf{p}}\hat a_{\mathbf{p}} \,-\, \mathbf{p}\,\hat b^\dagger_{-\mathbf{p}}\hat b_{-\mathbf{p}}\right) \;=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3\,2E_{\mathbf{p}}}\,\left(\mathbf{p}\,\hat a^\dagger_{\mathbf{p}}\hat a_{\mathbf{p}} \,+\, \mathbf{p}\,\hat b^\dagger_{\mathbf{p}}\hat b_{\mathbf{p}}\right). \quad \mathcal{Q.E.D.}$$

Problem **4**(a):

Given the commutation relations (S.19) through (S.22) and the free Hamiltonian (12), straightforward commutations show that

$$\hat H\hat a_{\mathbf{p}} \;=\; \hat a_{\mathbf{p}}(\hat H - E_{\mathbf{p}}),\quad \hat H\hat b_{\mathbf{p}} \;=\; \hat b_{\mathbf{p}}(\hat H - E_{\mathbf{p}}),\quad \hat H\hat a^\dagger_{\mathbf{p}} \;=\; \hat a^\dagger_{\mathbf{p}}(\hat H + E_{\mathbf{p}}),\quad \hat H\hat b^\dagger_{\mathbf{p}} \;=\; \hat b^\dagger_{\mathbf{p}}(\hat H + E_{\mathbf{p}}), \tag{S.34}$$

and therefore

$$\begin{aligned}
\hat a^H_{\mathbf{p}}(t) \;&=\; e^{it\hat H}\hat a^S_{\mathbf{p}}e^{-it\hat H} \;=\; e^{-itE_{\mathbf{p}}}\,\hat a^S_{\mathbf{p}}\,,\\
\hat b^H_{\mathbf{p}}(t) \;&=\; e^{it\hat H}\hat b^S_{\mathbf{p}}e^{-it\hat H} \;=\; e^{-itE_{\mathbf{p}}}\,\hat b^S_{\mathbf{p}}\,,\\
\hat a^{\dagger H}_{\mathbf{p}}(t) \;&=\; e^{it\hat H}\hat a^{\dagger S}_{\mathbf{p}}e^{-it\hat H} \;=\; e^{-itE_{\mathbf{p}}}\,\hat a^{\dagger S}_{\mathbf{p}}\,,\\
\hat b^{\dagger H}_{\mathbf{p}}(t) \;&=\; e^{it\hat H}\hat b^{\dagger S}_{\mathbf{p}}e^{-it\hat H} \;=\; e^{-itE_{\mathbf{p}}}\,\hat b^{\dagger S}_{\mathbf{p}}\,.
\end{aligned} \qquad \mathcal{Q.E.D.}$$

Problem **4**(b):

In the Schrödinger picture, eqs. (S.24) imply

$$\hat{\Phi}^S(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_\mathbf{p}}\, e^{i\mathbf{p}\mathbf{x}} \big(\hat{a}_\mathbf{p} + \hat{b}^\dagger_{-\mathbf{p}}\big)^S = \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_\mathbf{p}} \left(e^{i\mathbf{p}\mathbf{x}}\,\hat{a}_\mathbf{p} + e^{-i\mathbf{p}\mathbf{x}}\,\hat{b}^\dagger_\mathbf{p}\right)^S \tag{S.35}$$

and likewise

$$\hat{\Pi}^S(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_\mathbf{p}} \left(-iE_\mathbf{p} e^{i\mathbf{p}\mathbf{x}}\,\hat{a}_\mathbf{p} + iE_\mathbf{p} e^{-i\mathbf{p}\mathbf{x}}\,\hat{b}^\dagger_\mathbf{p}\right)^S. \tag{S.36}$$

In the Heisenberg picture, we use time-dependent creation and annihilation operators (16), hence

$$\begin{aligned}
\hat{\Phi}^H(\mathbf{x},t) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_\mathbf{p}} \left(e^{+i\mathbf{p}\mathbf{x}}\,\hat{a}^H_\mathbf{p}(t) + e^{-i\mathbf{p}\mathbf{x}}\,\hat{b}^{\dagger H}_\mathbf{p}(t)\right) \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_\mathbf{p}} \left(e^{+i\mathbf{p}\mathbf{x}} e^{-iE_\mathbf{p}t}\,\hat{a}^S_\mathbf{p} + e^{-i\mathbf{p}\mathbf{x}} e^{+iE_\mathbf{p}t}\,\hat{b}^{\dagger S}_\mathbf{p}\right) \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_\mathbf{p}} \left(e^{-ipx}\,\hat{a}^S_\mathbf{p} + e^{+ipx}\,\hat{b}^{\dagger S}_\mathbf{p}\right)_{p^0=E_\mathbf{p}}, \\
\hat{\Pi}^H(\mathbf{x},t) &= \cdots = \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_\mathbf{p}} \left(-ip^0\, e^{-ipx}\,\hat{a}^S_\mathbf{p} + ip^0\, e^{+ipx}\,\hat{b}^{\dagger S}_\mathbf{p}\right)_{p^0=E_\mathbf{p}} \\
&= \frac{\partial}{\partial t}\,\hat{\Phi}^H(\mathbf{x},t).
\end{aligned} \tag{S.37}$$

And the hermitian conjugate of the above gives

$$\begin{aligned}
\hat{\Phi}^{\dagger H}(\mathbf{x},t) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_\mathbf{p}} \left(e^{-ipx}\,\hat{a}^S_\mathbf{p} + e^{+ipx}\,\hat{b}^{\dagger S}_\mathbf{p}\right)^\dagger_{p^0=E_\mathbf{p}}, \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_\mathbf{p}} \left(e^{-ipx}\,\hat{b}^S_\mathbf{p} + e^{+ipx}\,\hat{a}^{\dagger S}_\mathbf{p}\right)_{p^0=E_\mathbf{p}}, \\
\hat{\Pi}^{\dagger H}(\mathbf{x},t) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_\mathbf{p}} \left(-ip^0\, e^{-ipx}\,\hat{a}^S_\mathbf{p} + ip^0\, e^{+ipx}\,\hat{b}^{\dagger S}_\mathbf{p}\right)^\dagger_{p^0=E_\mathbf{p}} \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_\mathbf{p}} \left(-ip^0\, e^{-ipx}\,\hat{b}^S_\mathbf{p} + ip^0\, e^{+ipx}\,\hat{a}^{\dagger S}_\mathbf{p}\right)_{p^0=E_\mathbf{p}} \\
&= \frac{\partial}{\partial t}\,\hat{\Phi}^{\dagger H}(\mathbf{x},t).
\end{aligned} \tag{S.38}$$

Problem **4**(c):

This is obvious from eqs. (17) and the commutation relations between the (Schrödinger-picture) creation and annihilation operators.

Problem **4**(d):

Again, we make use of eqs. (17) and the bosonic commutation relations, specifically eqs. (S.21) and (S.22):

$$\left[\hat{\Phi}^H(x), \hat{\Phi}^{\dagger H}(x')\right] =$$

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_\mathbf{p}} \int \frac{d^3\mathbf{p}'}{(2\pi)^3\, 2E_{\mathbf{p}'}} \left[\left(e^{-ipx}\, \hat{a}^S_\mathbf{p} + e^{+ipx}\, \hat{b}^{\dagger S}_\mathbf{p}\right), \left(e^{-ip'x'}\, \hat{b}^S_\mathbf{p} + e^{+ip'x'}\, \hat{a}^{\dagger S}_\mathbf{p}\right)\right]$$

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_\mathbf{p}} \int \frac{d^3\mathbf{p}'}{(2\pi)^3\, 2E_{\mathbf{p}'}} \Big(e^{-ipx+ip'x'} \times (2\pi)^3\, 2E_\mathbf{p}\, \delta^{(3)}(\mathbf{p}-\mathbf{p}')$$

$$+ \; e^{+ipx-ip'x'} \times -(2\pi)^3, 2E_\mathbf{p}\, \delta^{(3)}(\mathbf{p}-\mathbf{p}')\Big)$$

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3\, 2E_\mathbf{p}} \left(e^{-ip(x-x')} - e^{+ip(x-x')}\right)_{p_0=E_\mathbf{p}}$$

$$\equiv D(x-x') - D(x'-x).$$

Q.E.D.