Problem **1**(a):

As explained in class,

$$u(p,s)\;=\;\begin{pmatrix}\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,\xi_s\\ \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,\xi_s\end{pmatrix}\quad\Longrightarrow\quad \bar{u}(p,s)\;=\;\left(\xi_s^\dagger\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,,\xi_s^\dagger\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\right)\tag{S.1}$$

where ξ_s is the ordinary 3D 2–component spinor normalized to $\xi^\dagger\xi=1$ and therefore

$$\sum_s(\xi_s\xi_s^\dagger)\;=\;1\tag{S.2}$$

as a 2×2 matrix. Consequently, in 4×4 matrix notations, we have

$$\begin{aligned}\sum_s u(p,s)\bar{u}(p,s)\;&=\;\sum_s\begin{pmatrix}\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,(\xi_s\xi_s^\dagger)\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & \sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,(\xi_s\xi_s^\dagger)\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\\ \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,(\xi_s\xi_s^\dagger)\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & \sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,(\xi_s\xi_s^\dagger)\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\end{pmatrix}\\ &=\;\begin{pmatrix}\sqrt{E^2-(\mathbf{p}\boldsymbol{\sigma})^2} & (E-\mathbf{p}\boldsymbol{\sigma})\\ (E+\mathbf{p}\boldsymbol{\sigma}) & \sqrt{E^2-(\mathbf{p}\boldsymbol{\sigma})^2}\end{pmatrix}\\ &=\;\begin{pmatrix}m & (E-\mathbf{p}\boldsymbol{\sigma})\\ (E+\mathbf{p}\boldsymbol{\sigma}) & m\end{pmatrix}\;=\;m+\not{p}.\end{aligned}\tag{S.3}$$

Likewise, for the negative-frequency spinors we have

$$v(p,s)\;=\;\begin{pmatrix}+\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,\eta_s\\ -\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,\eta_s\end{pmatrix}\quad\Longrightarrow\quad \bar{v}(p,s)\;=\;\left(-\eta_s^\dagger\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,,+\eta_s^\dagger\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\right)\tag{S.4}$$

and therefore

$$\begin{aligned}\sum_s v(p,s)\bar{v}(p,s)\;&=\;\sum_s\begin{pmatrix}-\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,(\xi_s\xi_s^\dagger)\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & +\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\,(\xi_s\xi_s^\dagger)\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\\ +\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,(\xi_s\xi_s^\dagger)\sqrt{E+\mathbf{p}\boldsymbol{\sigma}} & -\sqrt{E+\mathbf{p}\boldsymbol{\sigma}}\,(\xi_s\xi_s^\dagger)\sqrt{E-\mathbf{p}\boldsymbol{\sigma}}\end{pmatrix}\\ &=\;\begin{pmatrix}-\sqrt{E^2-(\mathbf{p}\boldsymbol{\sigma})^2} & (E-\mathbf{p}\boldsymbol{\sigma})\\ (E+\mathbf{p}\boldsymbol{\sigma}) & -\sqrt{E^2-(\mathbf{p}\boldsymbol{\sigma})^2}\end{pmatrix}\\ &=\;\begin{pmatrix}-m & (E-\mathbf{p}\boldsymbol{\sigma})\\ (E+\mathbf{p}\boldsymbol{\sigma}) & -m\end{pmatrix}\;=\;-m+\not{p}.\end{aligned}\tag{S.5}$$

Problem **1**(b):

The constant spinors $u \equiv u(p,s)$ and $\bar{u}' \equiv \bar{u}(p',s')$ satisfy Dirac equations $\not{p}u = mu$ and $\bar{u}'\not{p}' = m\bar{u}'$. Applying both equations to the Dirac "sandwich" $\bar{u}'\gamma^\mu u$, we have

$$\bar{u}'\gamma^\mu u \;=\; \frac{1}{m}\,\bar{u}'\not{p}'\times\gamma^\mu u \;=\; \frac{1}{m}\,\bar{u}'\gamma^\mu\times\not{p}u \;=\; \frac{1}{2m}\,\bar{u}'(\not{p}'\gamma^\mu \,+\, \gamma^\mu\not{p})u. \tag{S.6}$$

Furthermore,

$$\begin{aligned}\not{p}'\gamma^\mu \,+\, \gamma^\mu\not{p} \;&\equiv\; p'_\nu\gamma^\nu\gamma^\mu \,+\, p_\nu\gamma^\mu\gamma^\nu \;=\; \tfrac{1}{2}(p'+p)_\nu\{\gamma^\mu,\gamma_\nu\} \,+\, \tfrac{1}{2}(p'-p)_\nu[\gamma^\nu,\gamma^\mu] \\ &=\; \tfrac{1}{2}(p'+p)_\nu\times 2g^{\mu\nu} \,+\, \tfrac{1}{2}(p'-p)_\nu\times 4iS^{\mu\nu}\end{aligned} \tag{S.7}$$

and therefore

$$\bar{u}'\gamma^\mu u \;=\; \frac{(p'+p)^\mu}{2m}\,\bar{u}'u \,+\, \frac{i(p'-p)_\nu}{m}\,\bar{u}'S^{\mu\nu}u. \tag{2}$$

$\mathcal{Q.E.D.}$

Problem **1**(c):

The negative-frequency spinors $v \equiv v(p,s)$ and $\bar{v}' \equiv \bar{v}(p',s')$ satisfy Dirac equations $\not{p}v = -mv$ and $\bar{v}'\not{p}' = -m\bar{v}'$. Consequently, proceeding exactly as above modulo signs, we have

$$\begin{aligned}\bar{u}'\gamma^\mu v \;&=\; \frac{(p'-p)^\mu}{2m}\,\bar{u}'v \,+\, \frac{i(p'+p)_\nu}{m}\,\bar{u}'S^{\mu\nu}v, \\ \bar{v}'\gamma^\mu u \;&=\; \frac{(-p'+p)^\mu}{2m}\,\bar{v}'u \,+\, \frac{i(-p'+p)_\nu}{m}\,\bar{v}'S^{\mu\nu}u, \\ \bar{v}'\gamma^\mu v \;&=\; \frac{(-p'-p)^\mu}{2m}\,\bar{v}'v \,+\, \frac{i(-p'+p)_\nu}{m}\,\bar{v}'S^{\mu\nu}v.\end{aligned} \tag{S.8}$$

Problem **2**(a):

First, let us rewrite the Lorentz algebra in terms of 3–vectors $\hat{\mathbf{J}}$ and $\hat{\mathbf{K}}$:

$$\left[\hat{J}^i,\hat{J}^j\right] \;=\; i\epsilon^{ij\ell}\,\hat{J}^\ell, \quad \left[\hat{J}^i,\hat{K}^j\right] \;=\; i\epsilon^{ij\ell}\,\hat{K}^\ell, \quad \left[\hat{K}^i,\hat{K}^j\right] \;=\; -i\epsilon^{ij\ell}\,\hat{J}^\ell. \tag{S.9}$$

Consequently, for the $\hat{\mathbf{J}}_\pm = \frac{1}{2}\big(\hat{\mathbf{J}} \pm i\hat{\mathbf{K}}\big)$, we have

$$\left[\hat{J}^i_\pm,\hat{J}^j_\pm\right] \;=\; \tfrac{i}{4}\epsilon^{ij\ell}\,\hat{J}^\ell \,\mp\, \tfrac{1}{4}\epsilon^{ij\ell}\,\hat{K}^\ell \,\mp\, \tfrac{1}{4}\epsilon^{ij\ell}\,\hat{K}^\ell \,+\, \tfrac{i}{4}\epsilon^{ij\ell}\,\hat{J}^\ell \;=\; i\epsilon^{ij\ell}\,\hat{J}^\ell_\pm$$

while

$$\left[\hat{J}^i_\pm, \hat{J}^j_\mp\right] \;=\; \tfrac{i}{4}\epsilon^{ij\ell}\,\hat{J}^\ell \;\mp\; \tfrac{1}{4}\epsilon^{ij\ell}\,\hat{K}^\ell \;\pm\; \tfrac{1}{4}\epsilon^{ij\ell}\,\hat{K}^\ell \;-\; \tfrac{i}{4}\epsilon^{ij\ell}\,\hat{J}^\ell \;=\; 0.$$

$\mathcal{Q.E.D.}$

Problem **2**(b):

First, note the hermiticity of the σ^μ matrices and the fact that any hermitian 2×2 matrix is a unique linear combination of the four σ_ν with real coefficients. Consequently,

$$\forall M:\; M\sigma^\mu M^\dagger \;=\; \sigma^\nu\, L_\nu^{\ \mu}(M) \Longrightarrow\; X'_\nu \;=\; L_\nu^{\ \mu}(M)X_\mu \tag{S.10}$$

for some real 4×4 matrix $L_\nu^{\ \mu}(M)$. Furthermore, for $M\in SL(2,\mathbf{C})$, *i.e.* for $\det(M)=1$, this $L_\nu^{\ \mu}(M)$ matrix defines a Lorentz transform for which $X'_\mu X'^\mu = X_\mu X^\mu$. To see this, we note that

$$\det\bigl(X_\mu\sigma^\mu\bigr) \;=\; \det\begin{pmatrix} X_0+X_3 & X_1-iX_2 \\ X_1+iX_2 & X_0-X_3 \end{pmatrix} \;=\; (X_0)^2-(X_3)^2-(X_1)^2-(X_2)^2 \;\equiv X^2 \tag{S.11}$$

and then calculate

$$X'^2 \;=\; \det\bigl(X'_\mu\sigma^\mu\bigr) \;=\; \det\bigl(M(X_\mu\sigma^\mu)M^\dagger\bigr) \;=\; \left|\det(M)\right|^2\times\det\bigl(X_\mu\sigma^\mu\bigr) \;=\; 1\times X^2. \tag{S.12}$$

Also, the Lorentz transform $X_\mu \to X'_\mu = L_\mu^{\ \nu}X_\nu$ is orthochronous because

$$L_0^{\ 0} \;=\; \tfrac{1}{2}\operatorname{tr}\bigl(\sigma^\nu L_\nu^{\ 0}\bigr) \;=\; \tfrac{1}{2}\operatorname{tr}\bigl(M\sigma^0 M^\dagger\bigr) \;=\; \tfrac{1}{2}\operatorname{tr}\bigl(MM^\dagger\bigr) \;>\; 0. \tag{S.13}$$

Problem **2**(b*):

The simplest proof the $L_\mu^{\ \nu}(M)$ is a proper Lorentz transform involves the group law (problem **2**(c) below) and the explicit examples of a pure rotation and a pure boost (problem **2**(d) below, eqs. (S.16) and (S.18)), both of which are manifestly proper.

For any $SL(2,\mathbf{C})$ matrix M we may decompose $M=HU$ where $H=\sqrt{MM^\dagger}$ is hermitian and $U=H^{-1}M$ is unitary. (Proof: $UU\dagger = H^{-1}MM^\dagger H^{-1} = H^{-1}H^2H^{-1} = 1$.) Furthermore, both H and U are unimodular ($\det(H)=\det(U)=1$), or in other words $H,U\in SL(2,\mathbf{C})$, which allows us to define two separate Lorentz transforms $L(H)$ and $L(U)$. According to the group law, together these two transform accomplish the $L(M)$ transform,

$$L(M) \;=\; L(H)\times L(U). \tag{S.14}$$

Now, H is hermitian, unimodular, and positive definite, hence it has a well-defined logarithm which is hermitian and traceless, $\operatorname{tr}(\log H)=0$. For the 2×2 matrices, this means $\log H=-\frac{1}{2}\mathbf{r}\boldsymbol{\sigma}$ for some real 3–vector $\mathbf{r}$, or equivalently $H=\exp\left(-\frac{1}{2}r\mathbf{n}\boldsymbol{\sigma}\right)$. As we shall see in eq. (S.18) below, this means that $L(H)$ is a pure Lorentz boost of rapidity r in the direction $\mathbf{n}$. This boost manifestly does not invert space or time, thus $L(U)$ is proper.

Likewise, U is unitary and unimodular, thus $U\in SU(2)$ and defines a pure rotation of space. Indeed, any $U\in SU(2)$ can be written as $U=\exp\left(-\frac{i}{2}\theta\mathbf{n}'\boldsymbol{\sigma}\right)$ for some angle θ and some axis $\mathbf{n}'$, and according to eq. (S.16) below $L(U)$ is indeed a pure space rotation by angle θ around axis $\mathbf{n}'$. Again, this rotation is proper — it does not invert space or time. Thus, $L(H)$ and $L(U)$ are both proper Lorentz transforms, hence their product $L(M)$ must also be proper. (Proof: $\det(L(M))=\det(L(H))\times\det(L(U))=+1$.) $\mathcal{Q.E.D.}$

Problem **2**(c):

$$\begin{aligned}\sigma_\lambda\,L^\lambda_{\ \mu}(M_2M_1) \;&=\; (M_2M_1)\sigma_\mu(M_2M_1)^\dagger \;=\; M_2\left(M_1\sigma_\mu M_1^\dagger \;=\; \sigma_\nu\,L^\nu_{\ \mu}(M_1)\right)M_2^\dagger\\ &=\;\left(M_2\,\sigma_\nu\,M_2^\dagger\right)\,L^\nu_{\ \mu}(M_1)\;=\;\sigma_\lambda\,L^\lambda_{\ \nu}(M_2)\,L^\nu_{\ \mu}(M_1)\end{aligned} \tag{S.15}$$

and hence $L^\lambda_{\ \mu}(M_2M_1)=L^\lambda_{\ \nu}(M_2)\,L^\nu_{\ \mu}(M_1)$ *i.e.*, $L(M_2M_1)=L(M_2)\,L(M_1)$. $\mathcal{Q.E.D.}$

Problem **2**(d):

For $M=\exp\bigl(-\frac{i}{2}\theta\,\mathbf{n}\boldsymbol{\sigma}\bigr)=\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\,\mathbf{n}\boldsymbol{\sigma}$ and $M^\dagger=M^{-1}=cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\,\mathbf{n}\boldsymbol{\sigma}$, $M\sigma^0M^\dagger=1=\sigma^0$, which means that $L(M)$ is merely a rotation of the 3d space. Specifically,

$$\begin{aligned}\boldsymbol{\sigma}\cdot\mathbf{x}' \;=\; M(\mathbf{x}\boldsymbol{\sigma})M^\dagger \;&=\; \cos^2\tfrac{\theta}{2}\;-\;\tfrac{i}{2}\sin\theta\,[\mathbf{n}\boldsymbol{\sigma},\mathbf{x}\boldsymbol{\sigma}]\;+\;\sin^2\tfrac{\theta}{2}\,(\mathbf{n}\boldsymbol{\sigma})(\mathbf{x}\boldsymbol{\sigma})(\mathbf{n}\boldsymbol{\sigma})\\ &=\;\cos^2\tfrac{\theta}{2}\;+\;\sin\theta\,(\mathbf{n}\times\mathbf{x})\cdot\boldsymbol{\sigma}\;+\;\sin^2\tfrac{\theta}{2}\,\bigl(2(\mathbf{n}\mathbf{x})(\mathbf{n}\boldsymbol{\sigma})-(\mathbf{x}\boldsymbol{\sigma})\bigr)\\ &=\;\boldsymbol{\sigma}\cdot\Bigl(\cos\theta\bigl(\mathbf{x}-\mathbf{n}(\mathbf{n}\mathbf{x})\bigr)\;+\;\sin\theta\,\mathbf{n}\times\mathbf{x}\;+\;\mathbf{n}(\mathbf{n}\mathbf{x})\Bigr)\,,\\ \text{thus}\quad\mathbf{x}'\;&=\;\cos\theta\bigl(\mathbf{x}-\mathbf{n}(\mathbf{n}\mathbf{x})\bigr)\;+\;\sin\theta\,\mathbf{n}\times\mathbf{x}\;+\;\mathbf{n}(\mathbf{n}\mathbf{x}),\end{aligned} \tag{S.16}$$

which indeed describes a rotation through angle θ around axis $\mathbf{n}$.

On the other hand, for $M = M^\dagger = \exp\bigl(-\frac{r}{2}\,\mathbf{n}\boldsymbol{\sigma}\bigr) = \cosh\frac{r}{2} - \sinh\frac{r}{2}\,\mathbf{n}\boldsymbol{\sigma}$,

$$\begin{aligned} M\bigl(x^\mu\sigma_\mu \equiv t - \mathbf{x}\boldsymbol{\sigma}\bigr)M^\dagger = {} & \cosh^2\tfrac{r}{2}\,(t-\mathbf{x}\boldsymbol{\sigma}) \;-\; \tfrac{1}{2}\sinh r\,\{\mathbf{n}\boldsymbol{\sigma}, t-\mathbf{x}\boldsymbol{\sigma}\} \\ & \qquad\qquad\qquad +\; \sinh^2\tfrac{r}{2}\,(\mathbf{n}\boldsymbol{\sigma})(t-\mathbf{x}\boldsymbol{\sigma})(\mathbf{n}\boldsymbol{\sigma}) \\ = {} & \cosh^2\tfrac{r}{2}\,(t-\mathbf{x}\boldsymbol{\sigma}) \;-\; \sinh r\,\bigl(t\,\mathbf{n}\boldsymbol{\sigma} - \mathbf{n}\mathbf{x}\bigr) \\ & \qquad\qquad\qquad +\; \sinh^2\tfrac{r}{2}\,\bigl(t - 2(\mathbf{n}\mathbf{x})(\mathbf{n}\boldsymbol{\sigma}) + (\mathbf{x}\boldsymbol{\sigma})\bigr) \\ = {} & \bigl(\cosh r\,t \;+\; \sinh r\,\mathbf{n}\mathbf{x}\bigr) \;-\; (\boldsymbol{\sigma}\mathbf{n})\bigl(\sinh r\,t \;+\; \cosh r\,\mathbf{n}\mathbf{x}\bigr) \\ & \qquad\qquad\qquad -\; \boldsymbol{\sigma}\cdot\bigl(\mathbf{x} - \mathbf{n}(\mathbf{n}\mathbf{x})\bigr), \end{aligned} \tag{S.17}$$

and therefore,

$$t' \;=\; (\cosh r)\,t \;+\; (\sinh r)\,\mathbf{n}\mathbf{x}, \quad \mathbf{x}' \;=\; \mathbf{n}\bigl((\sinh r)\,t \;+\; (\cosh r)\,\mathbf{n}\mathbf{x}\bigr) \;+\; \bigl(\mathbf{x} - \mathbf{n}(\mathbf{n}\mathbf{x})\bigr), \tag{S.18}$$

which is precisely the Lorentz boost of rapidity r in the direction $\mathbf{n}$. (The rapidity r is related to the usual parameters of a Lorentz boost according to $\beta = \tanh r$, $\gamma = \cosh r$, $\gamma\beta = \sinh r$. For several boosts in the same directions, the rapidities add up, $r_{\text{tot}} = r_1 + r_2 + \cdots$.) $\mathcal{Q.E.D.}$

Problem **2** (e):

For any Lie algebra equivalent to an angular momentum or its analytic continuation, the product of two doublets comprises a triplet and a singlet, $\mathbf{2}\otimes\mathbf{2} = \mathbf{3}\oplus\mathbf{1}$, or in (j) notations, $(\frac{1}{2})\otimes(\frac{1}{2}) = (1)\oplus(0)$. Furthermore, the triplet $\mathbf{4} = (1)$ is symmetric with respect to permutations of the two doublets while the singlet $\mathbf{1} = (0)$ is antisymmetric.

For two separate and independent types of angular momenta $\mathbf{J}_+$ and $\mathbf{J}_-$ we combine the j_+ quantum numbers independently of j_- and the j_- quantum numbers independently of j_+. Thus,

$$(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2}) \;=\; (1,1)\oplus(1,0)\oplus(0,1)\oplus(0,0). \tag{S.19}$$

Furthermore, the symmetric part of this product should be either symmetric with respect to both the j_+ and the j_- indices or antisymmetric with respect to both indices, thus

$$\bigl[(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2})\bigr]_{\text{sym}} \;=\; (1,1)\oplus(0,0). \tag{S.20}$$

Likewise, the antisymmetric part is either symmetric with respect to the j_+ but antisymmetric with respect to the j_- or the other way around, thus

$$\left[(\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2})\right]_{\text{antisym}} \;=\; (1,0)\oplus(0,1). \tag{S.21}$$

From the $SO(1.3)$ point of view, the $(\frac{1}{2},\frac{1}{2})$ multiplet is the Lorentz vector, hence the generic 2–index Lorentz tensor decomposes into irreducible multiplets according to eq. (S.19). Imposing symmetry conditions, we have eq. (S.20) for the *symmetric* 2–index tensor $T^{\mu\nu}=T^{\nu\mu}$ where the singlet $(0,0)$ corresponds to the trace T^{μ}_{μ} while the $(1,1)$ irreducible multiplet is the traceless symmetric tensor.

Likewise, the antisymmetric Lorentz tensor $F^{\mu\nu}=-F^{\nu\mu}$ decomposes according to eq. (S.21). Here, the irreducible components $(1,0)$ and $(0,1)$ are complex but conjugate to each other; individually, they describe antisymmetric tensors subject to complex duality conditions $\frac{1}{2}\epsilon^{\kappa\lambda\mu\nu}F_{\mu\nu}=\pm iF^{\kappa\lambda}$, *i.e.* $\mathbf{E}=\pm i\mathbf{B}$.

Problem **2**(f):

Without the $\gamma_\mu\Psi^\mu=0$ constraint, the spin-vector Ψ^μ_a is the tensor product or the Dirac spinor and the Lorentz vector, thus

$$\Big[(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})\Big]\otimes(\tfrac{1}{2}\,half) \;=\; (1,\tfrac{1}{2})\oplus(0,\tfrac{1}{2})\oplus(\tfrac{1}{2},1)\oplus(\tfrac{1}{2},0). \tag{S.22}$$

The constraint removes a Dirac spinor $\gamma_\mu\Psi^\mu \;\Rightarrow\; (\frac{1}{2},0)\oplus(0,\frac{1}{2})$, thus we are left with the $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ part for the Rarita–Schwinger spin-vector.

Problem **3**(a):

Let us evaluate $\hat\Phi(x'=Lx)$ according to eq. (9) but using $p'=Lp$ as an integration variable:

$$\begin{aligned}\hat\Phi(x'=Lx) \;&=\int\frac{d^3\mathbf{p}'}{(2\pi)^3\,2E_{\mathbf{p}'}}\,\Big[e^{-ip'x'}\,\hat a(p')\;+\;e^{+ip'x'}\,\hat a^\dagger(p')\Big]_{p'^0=E_{\mathbf{p}'}}\\ &=\int\frac{d^3\mathbf{p}}{(2\pi)^3\,2E_{\mathbf{p}}}\,\Big[e^{-ipx}\,\hat a(Lp)\;+\;e^{+ipx}\,\hat a^\dagger(Lp)\Big]_{p^0=E_{\mathbf{p}}}\end{aligned} \tag{S.23}$$

where the second equality follows from $p'x'=px$ and $\int\frac{d^3\mathbf{p}'}{2E_{\mathbf{p}'}}=\int\frac{d^3\mathbf{p}}{2E_{\mathbf{p}}}$. At the same time, eq. (11)

implies

$$\begin{aligned}\hat{\Phi}(Lx) \;&=\; \hat{\mathcal{D}}(L)\,\hat{\Phi}(x)\,\hat{\mathcal{D}}^\dagger(L)\\ &=\int\frac{d^3\mathbf{p}}{(2\pi)^3\,2E_{\mathbf{p}}}\,\Big[e^{-ipx}\,\hat{\mathcal{D}}(L)\,\hat{a}(p)\,\hat{\mathcal{D}}^\dagger(L)\;+\;e^{+ipx}\,\hat{\mathcal{D}}(L)\,\hat{a}^\dagger(p)\,\hat{\mathcal{D}}^\dagger(L)\Big]_{p^0=E_{\mathbf{p}}}\end{aligned}\tag{S.24}$$

Since eqs. (S.23) and (S.24) should agree for all x, the Fourier transforms of their respective right hand sides should agree for all p, thus

$$\hat{\mathcal{D}}(L)\,\hat{a}(p)\,\hat{\mathcal{D}}^\dagger(L)\;=\;\hat{a}(Lp),\qquad \hat{\mathcal{D}}(L)\,\hat{a}^\dagger(p)\,\hat{\mathcal{D}}^\dagger(L)\;=\;\hat{a}^\dagger(Lp).\tag{12}$$

Consequently

$$\begin{aligned}\mathcal{D}(L)\,|p\rangle \;&=\; \hat{\mathcal{D}}(L)\Big(\hat{a}^\dagger(p)\,|0\rangle\Big)\;=\;\hat{\mathcal{D}}(L)\,\hat{a}^\dagger(p)\Big(|0\rangle=\hat{\mathcal{D}}^\dagger(L)\,|0\rangle\Big)\\ &=\;\Big(\hat{\mathcal{D}}(L)\,\hat{a}^\dagger(p)\,\hat{\mathcal{D}}^\dagger(L)=\hat{a}^\dagger(Lp)\Big)\,|0\rangle\\ &=\;|Lp\rangle\,,\end{aligned}\tag{13.1}$$

and likewise

$$\begin{aligned}\mathcal{D}(L)\,|p_1,p_2\rangle \;&=\; \hat{\mathcal{D}}(L)\Big(\hat{a}^\dagger(p_1)\hat{a}^\dagger(p_2)\,|0\rangle\Big)\\ &=\;\hat{\mathcal{D}}(L)\,\hat{a}^\dagger(p_1)\hat{a}^\dagger(p_2)\Big(|0\rangle=\hat{\mathcal{D}}^\dagger(L)\,|0\rangle\Big)\\ &=\;\Big(\hat{\mathcal{D}}(L)\,\hat{a}^\dagger(p_1)\,\hat{\mathcal{D}}^\dagger(L)\Big)\Big(\hat{\mathcal{D}}(L)\,\hat{a}^\dagger(p_2)\,\hat{\mathcal{D}}^\dagger(L)\Big)\,|0\rangle\\ &=\;\Big(\hat{a}^\dagger(Lp_1)\Big)\Big(\hat{a}^\dagger(Lp_2)\Big)\,|0\rangle\\ &=\;|Lp_1,Lp_2\rangle\,,\end{aligned}\tag{13.2}$$

etc., etc. $\mathcal{Q.E.D.}$

Problem **3**(b):

Consider two sequential Lorentz transforms, first L_1 and then L_2. According to eq. (15) for the combined transform $L = L_2 L_1$,

$$\hat{\mathcal{D}}(L_2L_1)\,\hat{\phi}_A(x)\,\hat{\mathcal{D}}^\dagger(L_2L_1) \;=\; \sum_C M_A^{\ C}\big((L_2L_1)^{-1} = L_1^{-1}L_2^{-1}\big)\,\hat{\phi}_C(L_2L_1x). \tag{S.25}$$

On the other hand, applying eq. (15) first for the L_1 and then again for the L_2, we have

$$\begin{aligned}
\hat{\mathcal{D}}(L_2)\hat{\mathcal{D}}(L_1)\,\hat{\phi}_A(x)\,\Big(\hat{\mathcal{D}}(L_2)\hat{\mathcal{D}}(L_1)\Big)^\dagger \;&=\; \hat{\mathcal{D}}(L_2)\,\Big(\hat{\mathcal{D}}(L_1)\,\hat{\phi}_A(x)\,\hat{\mathcal{D}}^\dagger(L_1)\Big)\,\hat{\mathcal{D}}^\dagger(L_2) \\
&=\; \hat{\mathcal{D}}(L_2)\,\left(\sum_B M_A^{\ B}(L_1^{-1})\,\hat{\phi}_B(L_1x)\right)\,\hat{\mathcal{D}}^\dagger(L_2) \\
&=\; \sum_B M_A^{\ B}(L_1^{-1})\,\Big(\hat{\mathcal{D}}(L_2)\,\hat{\phi}_B(L_1x)\,\hat{\mathcal{D}}^\dagger(L_2)\Big) \\
&=\; \sum_{B,C} M_A^{\ B}(L_1^{-1})\,M_B^{\ C}(L_2^{-1})\,\hat{\phi}_C(L_2L_1x).
\end{aligned} \tag{S.26}$$

Clearly, consistency between eqs. (S.25) and (S.26) requires that the field representation $M_A^{\ C}(L)$ and the Fock-space representation $\hat{\mathcal{D}}(L)$ satisfy the same group law,

$$M_A^{\ C}(L_1^{-1}L_2^{-1}) \;=\; \sum_B M_A^{\ B}(L_1^{-1})\,M_B^{\ C}(L_2^{-1}) \iff \hat{\mathcal{D}}(L_2L_1) \;=\; \hat{\mathcal{D}}(L_2)\hat{\mathcal{D}}(L_1). \qquad \mathcal{Q.E.D.}$$

Problem **3**(c):

Reversing our derivation of eqs. (13.1–2), we see that eqs. (12) require

$$\begin{aligned}
\hat{\mathcal{D}}(L)\,\hat{a}^\dagger(p,s)\,\hat{\mathcal{D}}^\dagger(L) \;&=\sum_{s'} C_{s,s'}(L,p)\,\hat{a}^\dagger(Lp,s'), \\
\hat{\mathcal{D}}(L)\,\hat{b}^\dagger(p,s)\,\hat{\mathcal{D}}^\dagger(L) \;&=\sum_{s'} C_{s,s'}(L,p)\,\hat{b}^\dagger(Lp,s'),
\end{aligned} \tag{S.27}$$

and hence by hermitian conjugation,

$$\begin{aligned}
\hat{\mathcal{D}}(L)\,\hat{a}(p,s)\,\hat{\mathcal{D}}^\dagger(L) \;&=\sum_{s'} C^*_{s,s'}(L,p)\,\hat{a}(Lp,s'), \\
\hat{\mathcal{D}}(L)\,\hat{b}(p,s)\,\hat{\mathcal{D}}^\dagger(L) \;&=\sum_{s'} C^*_{s,s'}(L,p)\,\hat{b}(Lp,s').
\end{aligned} \tag{S.28}$$

Consequently, 'sandwiching' both sides of eq. (9) between $\hat{\mathcal{D}}(L)$ and $\hat{\mathcal{D}}^\dagger(L)$ operators gives us

$$\begin{aligned}
\hat{\mathcal{D}}(L)\,\hat{\phi}_A(x)\,\hat{\mathcal{D}}^\dagger(L) \;&=\; \int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_\mathbf{p}}\sum_s\Big[e^{-ipx}f_A(p,s)\,\Big(\hat{\mathcal{D}}(L)\,\hat{a}(p,s)\,\hat{\mathcal{D}}^\dagger(L)\Big)\\
&\qquad\qquad\qquad +\;e^{+ipx}h_A(p,s)\,\Big(\hat{\mathcal{D}}(L)\,\hat{b}^\dagger(p,s)\,\hat{\mathcal{D}}^\dagger(L)\Big)\Big]_{p^0=+E_\mathbf{p}}\\
&=\;\int\!\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_\mathbf{p}}\sum_s\Big[e^{-ipx}f_A(p,s)\sum_{s'}C^*_{s,s'}(L,p)\,\hat{a}(Lp,s')\\
&\qquad\qquad\qquad +\;e^{+ipx}h_A(p,s)\sum_{s'}C_{s,s'}(L,p)\,\hat{b}^\dagger(Lp,s')\Big]_{p^0=+E_\mathbf{p}}.
\end{aligned}\tag{S.29}$$

On the other hand, according to eq. (15)

$$\begin{aligned}
\hat{\mathcal{D}}(L)\,\hat{\phi}_A(x)\,\hat{\mathcal{D}}^\dagger(L) \;&=\; \sum_B M_A^{\;B}(L^{-1})\,\hat{\phi}_B(x'=Lx)\\
&=\;\sum_B M_A^{\;B}(L^{-1})\int\!\frac{d^3\mathbf{p}'}{(2\pi)^3\,2E_{\mathbf{p}'}}\sum_{s'}\begin{bmatrix} e^{-ip'x'}f_B(p',s')\,\hat{a}(p',s')\\ +\;e^{+ip'x'}h_B(p',s')\,\hat{b}^\dagger(p',s')\end{bmatrix}_{p'^0=E_{\mathbf{p}'}}\\
&=\;\sum_B M_A^{\;B}(L^{-1})\int\!\frac{d^3\mathbf{p}}{(2\pi)^3\,2E_\mathbf{p}}\sum_{s'}\begin{bmatrix} e^{-ipx}f_B(Lp,s')\,\hat{a}(Lp,s')\\ +\;e^{+ipx}h_B(Lp,s')\,\hat{b}^\dagger(Lp,s')\end{bmatrix}_{p^0=E_\mathbf{p}}
\end{aligned}\tag{S.30}$$

where as in eq. (S.23) $p'=Lp$ and $x'=Lx$. Comparing eqs. (S.29) and (S.30) and identifying coefficients of similar operators, we see that consistency between eqs. (15) and (17) requires

$$\begin{aligned}
\sum_s f_A(p,s)\,C_{s.\,s'}(L,p) \;&=\; \sum_B M_A^{\;B}(L^{-1})\,f_B(Lp,s'),\\
\sum_s h_A(p,s)\,C^*_{s.\,s'}(L,p) \;&=\; \sum_B M_A^{\;B}(L^{-1})\,h_B(Lp,s').
\end{aligned}\tag{S.31}$$

Finally, multiplying these equations by the $M_B^{\;A}(L)$ matrix, we arrive at eqs. (18). $\mathcal{Q.E.D.}$

As an example, consider a massive vector field $\hat{A}^\mu(x)$ which we have (in previous exercises) written in the form (16) where $\hat{b}^\dagger(p,\lambda)=\hat{a}^\dagger(p,\lambda)$ (due to hermiticity of $\hat{A}^\mu(x)$) and $f^\mu(p,\lambda)$ plays the role of $f_A(p,s)$ (as well as $h^*_A(p,s)$). Consequently, $f^\mu(p,\lambda)$ indeed transform according to eq. (18) where $M^\mu_{\;\nu}(L)=L^\mu_{\;\nu}$, as appropriate for the vector representation of the Lorentz group, while the matrix $C_{\lambda,\lambda'}$ rotates the helicity states into each other.

Similarly the Dirac spinors $u(p,s)$ and $v(p,s)$ also transform according to eqs. (18) where $M_{ab}(L)$ is the Dirac representation of the Lorentz group while the $C_{s,s'}$ matrices acts on the 3D spinors ξ_s and η_s used for construction of the Dirac spinors $u(p,s)$ and $v(p,s)$.