

Problem 1(a):

$\gamma^\mu \gamma^\nu = \pm \gamma^\nu \gamma^\mu$  where the sign is '+' for  $\mu = \nu$  and '-' otherwise. Hence for any product  $\Gamma$  of the  $\gamma$  matrices,  $\gamma^\mu \Gamma = (-1)^{n_\mu} \Gamma \gamma^\mu$  where  $n_\mu$  is the number of  $\gamma^{\nu \neq \mu}$  factors of  $\Gamma$ . For  $\Gamma = \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ ,  $n_\mu = 3$  for any  $\mu = 0, 1, 2, 3$ ; thus  $\gamma^\mu \gamma^5 = -\gamma^5 \gamma^\mu$ .

Problem 1(b):

First,

$$\begin{aligned}
 (\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger &= -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger = +i\gamma^3\gamma^2\gamma^1\gamma^0 \\
 &= +i((\gamma^3\gamma^2)\gamma^1)\gamma^0 = (-1)^3 i\gamma^0((\gamma^3\gamma^2)\gamma^1) \\
 &= (-1)^{3+2} i\gamma^0(\gamma^1(\gamma^3\gamma^2)) = (-1)^{3+2+1} i\gamma^0(\gamma^1(\gamma^2\gamma^3)) \\
 &= +i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv +\gamma^5.
 \end{aligned} \tag{S.1}$$

Second,

$$\begin{aligned}
 (\gamma^5)^2 &= \gamma^5(\gamma^5)^\dagger = (i\gamma^0\gamma^1\gamma^2\gamma^3)(i\gamma^3\gamma^2\gamma^1\gamma^0) = -\gamma^0\gamma^1\gamma^2(\gamma^3\gamma^3)\gamma^2\gamma^1\gamma^0 \\
 &= +\gamma^0\gamma^1(\gamma^2\gamma^2)\gamma^1\gamma^0 = -\gamma^0(\gamma^1\gamma^1)\gamma^0 = +\gamma^0\gamma^0 = +1.
 \end{aligned} \tag{S.2}$$

Problem 1(c):

Any four distinct  $\gamma^\kappa, \gamma^\lambda, \gamma^\mu, \gamma^\nu$  are  $\gamma^0, \gamma^1, \gamma^2, \gamma^3$  in some order. They all anticommute with each other, hence  $\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu = \epsilon^{\kappa\lambda\mu\nu}\gamma^0\gamma^1\gamma^2\gamma^3 \equiv -i\epsilon^{\kappa\lambda\mu\nu}\gamma^5$ . The rest is obvious.

Problem 1(d):

$$\begin{aligned}
 i\epsilon^{\kappa\lambda\mu\nu} \gamma_\kappa \gamma^5 &= \gamma_\kappa \gamma^{[\kappa\gamma^\lambda\gamma^\mu\gamma^\nu]} \\
 &= \frac{1}{4} \gamma_\kappa \left( \gamma^\kappa \gamma^{[\lambda\gamma^\mu\gamma^\nu]} - \gamma^{[\lambda]} \gamma^\kappa \gamma^{(\mu\gamma^\nu]} + \gamma^{[\lambda\gamma^\mu]} \gamma^\kappa \gamma^{(\nu]} - \gamma^{[\lambda\gamma^\mu\gamma^\nu]} \gamma^\kappa \right) \\
 &= \frac{1}{4} \left( 4\gamma^{[\lambda\gamma^\mu\gamma^\nu]} + 2\gamma^{[\lambda\gamma^\mu\gamma^\nu]} + 4g^{[\lambda\mu}\gamma^\nu] + 2\gamma^{[\nu\gamma^\mu\gamma^\lambda]} \right) \\
 &= \frac{1}{4}(4 + 2 + 0 - 2)\gamma^{[\lambda\gamma^\mu\gamma^\nu]} = \gamma^{[\lambda\gamma^\mu\gamma^\nu]}.
 \end{aligned} \tag{S.3}$$

Problem 1(e):

*Proof by inspection:* In the Weyl basis, the 16 matrices are

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^{[\mu}\gamma^{\nu]} = \begin{pmatrix} \sigma^{[\mu}\bar{\sigma}^{\nu]} & 0 \\ 0 & \bar{\sigma}^{[\mu}\sigma^{\nu]} \end{pmatrix}, \quad \gamma^5\gamma^\mu = \begin{pmatrix} 0 & -\sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}, \quad (\text{S.4})$$

and their linear independence is self-evident. Since there are only 16 independent  $4 \times 4$  matrices altogether, any such matrix  $\Gamma$  is a linear combination of the matrices (S.4). *Q.E.D.*

*Algebraic Proof:* Without making any assumption about the matrix form of the  $\gamma^\mu$  operators, let us consider the Clifford algebra  $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ . Because of these anticommutation relations, one may re-order any product of the  $\gamma$ 's as  $\pm\gamma^0 \dots \gamma^0 \gamma^1 \dots \gamma^1 \gamma^2 \dots \gamma^2 \gamma^3 \dots \gamma^3$  and then further simplify it to  $\pm(\gamma^0 \text{ or } 1) \times (\gamma^1 \text{ or } 1) \times (\gamma^2 \text{ or } 1) \times (\gamma^3 \text{ or } 1)$ . The net result is (up to a sign or  $\pm i$  factor) one of the 16 operators  $1, \gamma^\mu, \gamma^{[\mu}\gamma^{\nu]}, \gamma^5\gamma^\mu$  (*cf.* (d)) or  $\gamma^5$  (*cf.* (c)). Consequently, any operator  $\Gamma$  algebraically constructed of the  $\gamma^\mu$ 's is a linear combination of these 16 operators.

Incidentally, the algebraic argument explains why the  $\gamma^\mu$  (and hence all their products) should be realized as  $4 \times 4$  matrices since any lesser matrix size would not accommodate 16 independent products. That is, the  $\gamma$ 's are  $4 \times 4$  matrices in four spacetime dimensions; different dimensions call for different matrix sizes. Specifically, in spacetimes of *even* dimensions  $d$ , there are  $2^d$  independent products of the  $\gamma$  operators, so we need matrices of size  $2^{d/2} \times 2^{d/2}$ :  $2 \times 2$  in two dimensions,  $4 \times 4$  in four,  $8 \times 8$  in six,  $16 \times 16$  in eight,  $32 \times 32$  in ten, *etc., etc.*

In odd dimensions, there are only  $2^{d-1}$  independent operators because  $\gamma^{d+1} \equiv (i)\gamma^0\gamma^1 \dots \gamma^{d-1}$  — the analogue of the  $\gamma^5$  operator in 4d — commutes rather than anticommutes with all the  $\gamma^\mu$  and hence with the whole algebra. Consequently, one has two distinct representations of the Clifford algebra — one with  $\gamma^{d+1} = +1$  and one with  $\gamma^{d+1} = -1$  — but in each representation there are only  $2^{d-1}$  independent operator products, which call for the matrix size of  $2^{(d-1)/2} \times 2^{(d-1)/2}$ . For example, in three spacetime dimensions (two space, one time), can take  $(\gamma^0, \gamma^1, \gamma^2) = (\sigma_3, i\sigma_1, i\sigma_2)$  for  $\gamma^4 \equiv i\gamma^0\gamma^1\gamma^2 = +1$  or  $(\gamma^0, \gamma^1, \gamma^2) = (\sigma_3, i\sigma_1, -i\sigma_2)$  for  $\gamma^4 = -1$ , but in both cases we have  $2 \times 2$  matrices.

Problem 1(f):

Under a continuous Lorentz symmetry  $x \mapsto x' = Lx$ , the Dirac spinor field and its conjugate

transform according to

$$\Psi'(x') = M(L)\Psi(x = L^{-1}x'), \quad \bar{\Psi}'(x') = \bar{\Psi}(x = L^{-1}x')M^{-1}(L), \quad (\text{S.5})$$

hence any bilinear  $\bar{\Psi}\Gamma\Psi$  transforms according to

$$\bar{\Psi}'(x')\Gamma\Psi'(x') = \bar{\Psi}(x)\Gamma\Psi(x) \quad (\text{S.6})$$

where

$$\Gamma' = M^{-1}(L)\Gamma M(L). \quad (\text{S.7})$$

Obviously, for  $\Gamma = 1$ ,  $\Gamma' = M^{-1}M = 1$ . According to homework set #5 (problem 3(b)), for  $\Gamma = \gamma^\mu$ ,  $\Gamma' = M^{-1}\gamma^\mu M = L^\mu_\nu \gamma^\nu$ . Similarly,  $M^{-1}\gamma^\mu \gamma^\nu M = (M^{-1}\gamma^\mu M)(M^{-1}\gamma^\nu M) = L^\mu_\kappa \gamma^\kappa \times L^\nu_\lambda \gamma^\lambda$  and hence for  $\Gamma = \gamma^{[\mu}\gamma^{\nu]}$ ,  $\Gamma' = L^\mu_\kappa L^\nu_\lambda \gamma^{[\kappa}\gamma^{\lambda]}$ . Consequently,

$$S'(x') = S(x), \quad V'^\mu(x') = L^\mu_\nu V^\nu(x), \quad T'^{\mu\nu}(x') = L^\mu_\kappa L^\nu_\lambda T^{\kappa\lambda}(x), \quad (\text{S.8})$$

which makes  $S$  a Lorentz scalar,  $V^\mu$  a Lorentz vector and  $T^{\mu\nu}$  a Lorentz tensor (with two antisymmetric indices).

The  $\gamma^5$  matrix commutes with even products of the  $\gamma^\mu$  matrices such as  $\gamma^\mu \gamma^\nu$ , hence it commutes with all  $S^{\mu\nu}$  and therefore with  $M(L) = \exp(-\frac{i}{2}\theta_{\mu\nu}S^{\mu\nu})$ . Consequently, for  $\Gamma = \gamma^5$ ,  $\Gamma' = M^{-1}\gamma^5 M = \gamma^5$  while for  $\Gamma = \gamma^5 \gamma^\mu$ ,  $\Gamma' = M^{-1}\gamma^5 \gamma^\mu M = \gamma^5 M^{-1}\gamma^\mu M = \gamma^5(L^\mu_\nu \gamma^\nu) = L^\mu_\nu (\gamma^5 \gamma^\nu)$ . Therefore,

$$P'(x') = P(x), \quad A'^\mu(x') = L^\mu_\nu A^\nu(x), \quad (\text{S.9})$$

which makes  $P$  a Lorentz scalar and  $A$  a Lorentz vector. *Q.E.D.*

Problem 2(a):

Given  $\Psi'(\mathbf{x}', t) = \pm\gamma^0\Psi(\mathbf{x} = -\mathbf{x}', t' = t)$ , we have

$$\begin{aligned} (i\cancel{\partial}' - m)\Psi'(x') &\equiv (i\gamma^0\partial_0 + i\vec{\gamma} \cdot \nabla' - m)(\pm\gamma^0)\Psi(\mathbf{x}', t) = (\pm\gamma^0)(i\gamma^0\partial_0 - i\vec{\gamma} \cdot \nabla' - m)\Psi(\mathbf{x}', t) \\ &= (\pm\gamma^0)(i\gamma^0\partial_0 + i\vec{\gamma} \cdot \nabla - m)\Psi(-\mathbf{x}, t) \\ &\equiv (\pm\gamma^0)(i\cancel{\partial} - m)\Psi\Big|_{x'}. \end{aligned} \quad (\text{S.10})$$

Problem 2(b):

Parity properties of the Dirac bilinears (1) follow from the commutation relations of the 16 operators (1e) with the  $\gamma^0$ . It is easy to verify that the 1,  $\gamma^0$ ,  $\gamma^{[i}\gamma^{j]}$  and  $\gamma^5\gamma^i$  commute with the  $\gamma^0$  while the  $\gamma^i$ ,  $\gamma^0\gamma^i$ ,  $\gamma^5\gamma^0$  and  $\gamma^5$  anticommute with the  $\gamma^0$ . Consequently,

- the  $S$ ,  $V^0$ ,  $T^{ij}$  and  $A^i$  remain invariant under parity, while
- the  $V^i$ ,  $T^{0i}$ ,  $A^0$  and  $P$  change their signs.

In three-dimensional terms, this means that  $S$  and  $V^0$  are true scalars,  $P$  and  $A^0$  are pseudoscalars,  $\mathbf{V}$  is a true or polar vector,  $\mathbf{A}$  is a pseudovector or axial vector, and the tensor  $T$  contains one true vector  $T^{0i}$  and one axial vector  $\frac{1}{2}\epsilon^{ijk}T^{jk}$ . In space-time terms, we call  $S$  a (Lorentz) (true) scalar,  $P$  a (Lorentz) pseudoscalar,  $V^\mu$  a (Lorentz) (true) vector and  $A^\mu$  an (Lorentz) axial vector. Pedantically speaking,  $T^{\mu\nu}$  is a Lorentz true tensor while  $\tilde{T}^{\kappa\lambda} \equiv \frac{1}{2}\epsilon^{\kappa\lambda\mu\nu}T_{\mu\nu}$  is a Lorentz pseudotensor, but few people are that pedantic.

Problem 3(a):

Charge conjugation acts on a Dirac spinor field according to  $\hat{C}\hat{\Psi}\hat{C} = \pm\gamma^2\hat{\Psi}^*$ . Consequently,

$$\hat{C}\hat{\Psi}\hat{C} = (\hat{C}\hat{\Psi}\hat{C})^\dagger\gamma^0 = \mp\Psi^\top\gamma^2\gamma^0 \quad (\text{S.11})$$

and hence

$$\hat{C}\hat{\Psi}\Gamma\hat{\Psi}\hat{C} = \hat{C}\hat{\Psi}\hat{C}\Gamma\hat{C}\hat{\Psi}\hat{C} = -\hat{\Psi}^\top\gamma^2\gamma^0\Gamma\gamma^2\hat{\Psi}^* = +\hat{\Psi}^\dagger(\gamma^2\gamma^0\Gamma\gamma^2)^\top\hat{\Psi} = \hat{\Psi}\gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2\hat{\Psi} \equiv \hat{\Psi}\Gamma^c\hat{\Psi}. \quad (\text{S.12})$$

Problem 3(b):

By inspection,  $\mathbf{1}^c \equiv \gamma^0\gamma^2\gamma^0\gamma^2 = +\mathbf{1}$ . The  $\gamma_5$  matrix is symmetric and commutes with the  $\gamma^0\gamma^2$ , hence  $\gamma_5^c = +\gamma_5$ . Among the four  $\gamma_\mu$  matrices, the  $\gamma_1$  and  $\gamma_3$  are anti-symmetric and commute with the  $\gamma^0\gamma^2$  while the  $\gamma_0$  and  $\gamma_2$  are symmetric but anti-commute with the  $\gamma^0\gamma^2$ ; hence, for all four  $\gamma_\mu$ ,  $\gamma_\mu^c = -\gamma_\mu$ . Finally, because of the transposition involved,  $(\gamma_\mu\gamma_\nu)^c = \gamma_\nu^c\gamma_\mu^c = +\gamma_\nu\gamma_\mu$ , thus  $\gamma_{\mu\nu}^c = +\gamma_{\nu\mu} = -\gamma_{\mu\nu}$ . Likewise,  $(\gamma_5\gamma_\mu)^c = \gamma_\mu^c\gamma_5^c = -\gamma_\mu\gamma_5 = +\gamma_5\gamma_\mu$ .

To summarize, the scalar  $S$ , the pseudoscalar  $P$  and the axial vector  $A_\mu$  are C-even while the vector  $V_\mu$  and the tensor  $T_{\mu\nu}$  are C-odd.

Problem 4(a):

The overall statistics of the operator product  $\hat{B}\hat{C}$  corresponds to  $(-1)^{A(BC)} = (-1)^{AB}(-1)^{AC}$ . Therefore,

$$\begin{aligned}
[\hat{A}, \hat{B}\hat{C}] &\stackrel{\text{def}}{=} \hat{A}\hat{B}\hat{C} - (-1)^{AB}(-1)^{AC}\hat{B}\hat{C}\hat{A} \\
&= \left(\hat{A}\hat{B} - (-1)^{AB}\hat{B}\hat{A}\right)\hat{C} + (-1)^{AB}\hat{B}\left(\hat{A}\hat{C} - (-1)^{AC}\hat{C}\hat{A}\right) \\
&= [\hat{A}, \hat{B}]\hat{C} + (-1)^{AB}\hat{B}[\hat{A}, \hat{C}].
\end{aligned} \tag{S.13}$$

Likewise,

$$\begin{aligned}
[\hat{A}\hat{B}, \hat{C}] &= \hat{A}\hat{B}\hat{C} - (-1)^{AC}(-1)^{BC}\hat{C}\hat{A}\hat{B} \\
&= \hat{A}\left(\hat{B}\hat{C} - (-1)^{BC}\hat{C}\hat{B}\right) + (-1)^{BC}\left(\hat{A}\hat{C} - (-1)^{AC}\hat{C}\hat{A}\right)\hat{B} \\
&= \hat{A}[\hat{B}, \hat{C}] + (-1)^{BC}[\hat{A}, \hat{C}]\hat{B}.
\end{aligned} \tag{S.14}$$

Problem 4(b):

$$\begin{aligned}
[\hat{A}\hat{B}, \hat{C}\hat{D}] &= \hat{A}[\hat{B}, \hat{C}\hat{D}] + (-1)^{BC}(-1)^{BD}[\hat{A}, \hat{C}\hat{D}]\hat{B} \\
&= \hat{A}[\hat{B}, \hat{C}]\hat{D} + (-1)^{BC}\hat{A}\hat{C}[\hat{B}, \hat{D}] \\
&\quad + (-1)^{BC}(-1)^{BD}[\hat{A}, \hat{C}]\hat{D}\hat{B} + (-1)^{AC}(-1)^{BC}(-1)^{BD}\hat{C}[\hat{A}, \hat{D}]\hat{B}.
\end{aligned} \tag{S.15}$$

Problem 4(c):

$$\begin{aligned}
(-1)^{CA}[\hat{A}, [\hat{B}, \hat{C}]] &= (-1)^{CA}\hat{A}\hat{B}\hat{C} - (-1)^{BC}(-1)^{CA}\hat{A}\hat{C}\hat{B} \\
&\quad - (-1)^{AB}\hat{B}\hat{C}\hat{A} + (-1)^{AB}(-1)^{BC}\hat{C}\hat{B}\hat{A}, \\
(-1)^{BC}[\hat{C}, [\hat{A}, \hat{B}]] &= (-1)^{BC}\hat{C}\hat{A}\hat{B} - (-1)^{AB}(-1)^{BC}\hat{C}\hat{B}\hat{A} \\
&\quad - (-1)^{CA}\hat{A}\hat{B}\hat{C} + (-1)^{CA}(-1)^{AB}\hat{B}\hat{A}\hat{C}, \\
(-1)^{AB}[\hat{B}, [\hat{C}, \hat{A}]] &= (-1)^{AB}\hat{B}\hat{C}\hat{A} - (-1)^{CA}(-1)^{AB}\hat{B}\hat{A}\hat{C} \\
&\quad - (-1)^{BC}\hat{C}\hat{A}\hat{B} + (-1)^{BC}(-1)^{CA}\hat{A}\hat{C}\hat{B}.
\end{aligned} \tag{S.16}$$

Upon adding these three equations together, their right hand sides cancel out while the left hand sides add up to the Jacobi identity (4).

Problem 5(a):

Using the Leibniz rules (S.13) and (S.14) and the anticommutation relations (6), the calculation is straightforward.

$$\begin{aligned} [\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger] &= \delta_{\beta\gamma} \hat{a}_\alpha^\dagger, \\ [\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\delta] &= -\delta_{\alpha\delta} \hat{a}_\beta, \\ [\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger \hat{a}_\delta] &= \delta_{\beta\gamma} \hat{a}_\alpha^\dagger \hat{a}_\delta - \delta_{\alpha\delta} \hat{a}_\gamma^\dagger \hat{a}_\beta. \end{aligned} \tag{S.17}$$

Problem 5(b):

According to eq. (S.17), the commutator  $[\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger \hat{a}_\delta]$  has exactly the same form as its bosonic counterpart. Hence, the proof of  $[\hat{A}, \hat{B}] = \hat{C}$  proceeds exactly as in the bosonic case, *cf.* homework set #2 (problem 3(b)).

Problem 5(c):

Using the Leibniz rules and eqs. (S.17),

$$[\hat{a}_\mu^\dagger \hat{a}_\nu, \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta] = \delta_{\nu\alpha} \hat{a}_\mu^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta + \delta_{\nu\beta} \hat{a}_\mu^\dagger \hat{a}_\alpha^\dagger \hat{a}_\gamma \hat{a}_\delta - \delta_{\mu\gamma} \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\nu \hat{a}_\delta - \delta_{\mu\delta} \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\nu. \tag{S.18}$$

Problem 5(d):

Again, we have a fermionic analogue to the bosonic second-quantized operators we studied in homework set #3 (problem 1(d)). Given eqs. (7) and (S.18) (in which we exchange  $\gamma \leftrightarrow \delta$ ), we have

$$\begin{aligned} [\hat{A}, \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\delta \hat{a}_\gamma] &= \sum_{\mu, \nu} \langle \mu | \hat{A}_1 | \nu \rangle [\hat{a}_\mu^\dagger \hat{a}_\nu, \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\delta \hat{a}_\gamma] \\ &= \sum_{\mu} \langle \mu | \hat{A}_1 | \alpha \rangle \hat{a}_\mu^\dagger \hat{a}_\beta^\dagger \hat{a}_\delta \hat{a}_\gamma + \sum_{\mu} \langle \mu | \hat{A}_1 | \beta \rangle \hat{a}_\alpha^\dagger \hat{a}_\mu^\dagger \hat{a}_\delta \hat{a}_\gamma \\ &\quad - \sum_{\nu} \langle \delta | \hat{A}_1 | \nu \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\nu \hat{a}_\gamma - \sum_{\nu} \langle \gamma | \hat{A}_1 | \nu \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\delta \hat{a}_\nu \end{aligned} \tag{S.19}$$

and consequently, in light of eq. (8),

$$\begin{aligned}
[\hat{A}, \hat{B}] &= \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha \otimes \beta | \hat{B}_2 | \gamma \otimes \delta \rangle \left[ \sum_{\mu} \langle \mu | \hat{A}_1 | \alpha \rangle \hat{a}_{\mu}^{\dagger} \hat{a}_{\beta}^{\dagger} \hat{a}_{\delta} \hat{a}_{\gamma} + \sum_{\mu} \langle \mu | \hat{A}_1 | \beta \rangle \hat{a}_{\alpha}^{\dagger} \hat{a}_{\mu}^{\dagger} \hat{a}_{\delta} \hat{a}_{\gamma} \right. \\
&\quad \left. - \sum_{\nu} \langle \delta | \hat{A}_1 | \nu \rangle \hat{a}_{\alpha}^{\dagger} \hat{a}_{\beta}^{\dagger} \hat{a}_{\nu} \hat{a}_{\gamma} - \sum_{\nu} \langle \gamma | \hat{A}_1 | \nu \rangle \hat{a}_{\alpha}^{\dagger} \hat{a}_{\beta}^{\dagger} \hat{a}_{\delta} \hat{a}_{\nu} \right] \\
&= \sum_{\mu, \beta, \gamma, \delta} \langle \mu \otimes \beta | \hat{A}_1(1^{\text{st}}) \hat{B}_2 | \gamma \otimes \delta \rangle \hat{a}_{\mu}^{\dagger} \hat{a}_{\beta}^{\dagger} \hat{a}_{\delta} \hat{a}_{\gamma} \\
&\quad + \sum_{\alpha, \mu, \gamma, \delta} \langle \alpha \otimes \mu | \hat{A}_1(2^{\text{nd}}) \hat{B}_2 | \gamma \otimes \delta \rangle \hat{a}_{\alpha}^{\dagger} \hat{a}_{\mu}^{\dagger} \hat{a}_{\delta} \hat{a}_{\gamma} \\
&\quad - \sum_{\alpha, \beta, \gamma, \nu} \langle \alpha \otimes \beta | \hat{B}_2 \hat{A}_1(2^{\text{nd}}) | \gamma \otimes \nu \rangle \hat{a}_{\alpha}^{\dagger} \hat{a}_{\beta}^{\dagger} \hat{a}_{\nu} \hat{a}_{\gamma} \\
&\quad - \sum_{\alpha, \beta, \nu, \delta} \langle \alpha \otimes \beta | \hat{B}_2 \hat{A}_1(1^{\text{st}}) | \nu \otimes \delta \rangle \hat{a}_{\alpha}^{\dagger} \hat{a}_{\beta}^{\dagger} \hat{a}_{\delta} \hat{a}_{\nu}
\end{aligned}$$

⟨renaming indices⟩

$$\begin{aligned}
&= \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha \otimes \beta | \left[ (A_1(1^{\text{st}}) + A_1(2^{\text{nd}})), \hat{B}_2 \right] | \gamma \otimes \delta \rangle \times \hat{a}_{\alpha}^{\dagger} \hat{a}_{\beta}^{\dagger} \hat{a}_{\delta} \hat{a}_{\gamma} \\
&= \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha \otimes \beta | \hat{C}_2 | \gamma \otimes \delta \rangle \times \hat{a}_{\alpha}^{\dagger} \hat{a}_{\beta}^{\dagger} \hat{a}_{\delta} \hat{a}_{\gamma} \equiv \hat{C}.
\end{aligned}$$

*Q. E. D.*