

First, we need a Lemma:

Consider a decay or a scattering process producing three final state particles. For any such process, the phase space of the final states is

$$d\mathcal{P} = \frac{d^3\mathbf{p}_1}{(2\pi)^3(2E_1)} \frac{d^3\mathbf{p}_2}{(2\pi)^3(2E_2)} \frac{d^3\mathbf{p}_3}{(2\pi)^3(2E_3)} (2\pi)^3 \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) (2\pi) \delta(E_1 + E_2 + E_3 - E_{\text{tot}}) \quad (\text{S.1})$$

in the center-of-mass frame. In this lemma, we shall see that regardless of the three particles' masses, the phase space (S.1) can be written as

$$d\mathcal{P} = \frac{d^3\Omega}{256\pi^5} \times dE_1 dE_2 dE_3 \delta(E_1 + E_2 + E_3 - E_{\text{tot}}) \quad (\text{S.2})$$

where $d^3\Omega$ refers to three angular variables parameterizing the directions of the three particles relative to some external frame but not affecting the angles between the three momenta. For example, one may use two angles to describe the orientation of the decay plane (the three momenta are coplanar, $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0$) and one more angle to fix the direction of *e.g.*, \mathbf{p}_1 in that plane. Altogether, $\int d^3\Omega = 4\pi \times 2\pi = 8\pi^2$.

We begin the proof with the obvious step of eliminating the momentum conservation delta-function, thus

$$d\mathcal{P} = \frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2}{256\pi^5} \frac{\delta(E_1 + E_2 + E_3 - E_{\text{tot}})}{E_1 E_2 E_3} \Bigg|_{\mathbf{p}_3 = -(\mathbf{p}_1 + \mathbf{p}_2)}. \quad (\text{S.3})$$

Next, we use spherical coordinates for the two remaining momenta,

$$d^3\mathbf{p}_1 = p_1^2 dp_1 d^2\Omega_1, \quad d^3\mathbf{p}_2 = p_2^2 dp_2 d^2\Omega_2, \quad (\text{S.4})$$

and then replace the $d^2\Omega_2$ describing the direction of the second particle's momentum relative to the fixed external frame with

$$d^2\Omega_2^{(1)} = d\theta_{12} \sin\theta_{12} d\phi_2^{(1)}$$

describing the same direction of \mathbf{p}_2 relative to the frame centered on the \mathbf{p}_1 . Consequently,

$$d^2\Omega_1 d^2\Omega_2 = d^2\Omega_1 d^2\Omega_2^{(1)} = \left[d^2\Omega_1 d\phi_2^{(1)} \right] d\theta_{12} \sin\theta_{12} \equiv d^3\Omega \times d(\cos\theta_{12}) \quad (\text{S.5})$$

and hence

$$d\mathcal{P} = \frac{d^3\Omega}{256\pi^5} \times \frac{p_1^2 p_2^2}{E_1 E_2 E_3} dp_1 dp_2 d(\cos\theta_{12}) \delta(E_1 + E_2 + E_3 - E_{\text{tot}}) \Big|_{\mathbf{p}_3 = -(\mathbf{p}_1 + \mathbf{p}_2)}. \quad (\text{S.6})$$

Next, we use the cosine theorem

$$p_3^2 = (\mathbf{p}_1 + \mathbf{p}_2)^2 = p_1^2 + p_2^2 + 2p_1 p_2 \cos\theta_{12}$$

which gives

$$d(\cos\theta_{12}) = \frac{p_3 dp_3}{p_1 p_2}$$

(for fixed p_1, p_2) and therefore

$$d\mathcal{P} = \frac{d^3\Omega}{256\pi^5} \times \frac{p_1 p_2 p_3}{E_1 E_2 E_3} dp_1 dp_2 dp_3 \delta(E_1 + E_2 + E_3 - E_{\text{tot}}). \quad (\text{S.7})$$

Finally, we notice that for a relativistic particle of any mass, $pdp = EdE$ and hence (S.2).

It remains to determine the limits of kinematically allowed ways to distribute the net energy E_{tot} of the process among the three final particles. Such limits follow from the triangle inequalities for the three momenta,

$$p_1 \leq p_2 + p_3, \quad p_2 \leq p_1 + p_3, \quad p_3 \leq p_1 + p_2, \quad (\text{S.8})$$

which look simple but produce rather complicated inequalities for the energies. However, when all three final particles are massless, the kinematic restrictions become simply

$$E_1, E_2, E_3 \leq \frac{1}{2}E_{\text{tot}}, \quad E_1 + E_2 + E_3 = E_{\text{tot}}. \quad (\text{S.9})$$

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We are now ready to calculate the muon decay rate and the electron's spectrum. According to the general rules of decay

$$d\Gamma = \frac{|\mathcal{M}|^2}{2M} d\mathcal{P}, \quad (\text{S.10})$$

thus in light of eqs. (2) and (S.2),

$$d\Gamma(\mu^- \rightarrow e^- \nu_\mu \bar{\nu}_e) = \frac{G_F^2}{8\pi^5 M_\mu} (p_\mu \cdot p_{\bar{\nu}})(p_e \cdot p_\nu) \times dE_e dE_\nu dE_{\bar{\nu}} d^3\Omega \delta(E_e + E_\nu + E_{\bar{\nu}} - M_\mu). \quad (\text{S.11})$$

In the muon's frame,

$$(p_\mu \cdot p_{\bar{\nu}}) = M_\mu E_{\bar{\nu}} \quad (\text{S.12})$$

while

$$(p_e \cdot p_e) = E_e E_\nu - p_e p_\nu \cos \theta_{e\nu} = E_e E_\nu + \frac{1}{2} p_e^2 + \frac{1}{2} p_\nu^2 - \frac{1}{2} p_{\bar{\nu}}^2; \quad (\text{S.13})$$

neglecting the electron and neutrino's masses, we may rewrite (S.13) as

$$\frac{1}{2}(E_e + E_\nu)^2 - \frac{1}{2}E_{\bar{\nu}}^2 = \frac{1}{2}M_\mu(M_\mu - 2E_{\bar{\nu}}). \quad (\text{S.14})$$

Consequently,

$$d\Gamma(\mu^- \rightarrow e^- \nu_\mu \bar{\nu}_e) = \frac{G_F^2}{16\pi^5} M_\mu E_{\bar{\nu}} (M_\mu - 2E_{\bar{\nu}}) \times dE_e dE_\nu dE_{\bar{\nu}} d^3\Omega \delta(E_e + E_\nu + E_{\bar{\nu}} - M_\mu). \quad (\text{S.15})$$

At this point we are ready to integrate over the final-state variables. In light of $\int d^3\Omega = 8\pi^2$ and the kinematic limits (S.9), we immediately obtain

$$\begin{aligned} \Gamma(\mu^- \rightarrow e^- \nu_\mu \bar{\nu}_e) &= \frac{G_F^2 M_\mu}{2\pi^3} \int_0^{\frac{1}{2}M_\mu} \int_0^{\frac{1}{2}M_\mu} \int_0^{\frac{1}{2}M_\mu} dE_e dE_{\bar{\nu}} dE_\nu E_{\bar{\nu}} (M_\mu - 2E_{\bar{\nu}}) \delta(E_e + E_\nu + E_{\bar{\nu}} - M_\mu) \\ &= \frac{G_F^2 M_\mu}{2\pi^3} \int_0^{\frac{1}{2}M_\mu} dE_e \int_{\frac{1}{2}M_\mu - E_e}^{\frac{1}{2}M_\mu} dE_{\bar{\nu}} E_{\bar{\nu}} (M_\mu - 2E_{\bar{\nu}}) \\ &= \frac{G_F^2 M_\mu}{2\pi^3} \int_0^{\frac{1}{2}M_\mu} dE_e E_e^2 \left(\frac{1}{2}M_\mu - \frac{2}{3}E_e\right). \end{aligned} \quad (\text{S.16})$$

In other words, the partial muon decay rate with respect to the final electron's energy is given

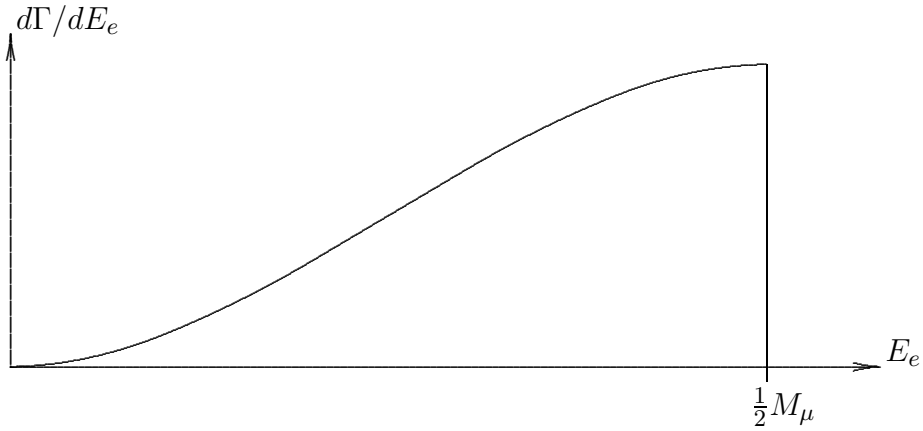
by

$$\frac{d\Gamma}{dE_e} = \frac{G_F^2 M_\mu}{12\pi^3} \times E_e^2 (3M_\mu - 4E_e) \quad (\text{S.17})$$

or rather

$$\frac{d\Gamma}{dE_e} \approx \begin{cases} \frac{G_F^2}{12\pi^3} M_\mu E_e^2 (3M_\mu - 4E_e) & \text{for } E_e < \frac{1}{2}M_\mu, \\ 0 & \text{for } E_e > \frac{1}{2}M_\mu. \end{cases} \quad (\text{S.18})$$

Graphically,



Note how this curve smoothly reaches its maximum at $E_e = \frac{1}{2}M_\mu$ and then abruptly falls down to zero.

It remains to calculate the total decay rate of the muon by integrating the partial rate (S.18) over the electron's energy. The result is

$$\Gamma_{\text{tot}}(\mu \rightarrow e\nu\bar{\nu}) = \frac{G_F^2 M_\mu^5}{192\pi^3}. \quad (\text{S.19})$$