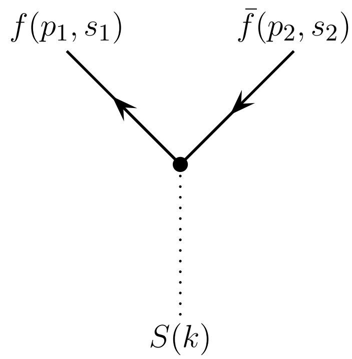
Problem **1**:

At the tree level, the decay $S \to f + \bar{f}$ follows from a single Feynman diagram



$$\text{hence} \quad i\mathcal{M} \;=\; \bar{u}(p_1, s_1)(-ig)v(p_2, s_2). \tag{S.1}$$

Note that this amplitude refers to specific spin states of the final fermion and antifermion. For the purpose of calculating the net (*i.e.*, un-polarized) decay rate, we need to sum the $|\mathcal{M}|^2$ over the final state spins. Thus

$$\begin{aligned}
\sum_{s_1,s_2} |\mathcal{M}|^2 \;&=\; g^2 \sum_{s_1,s_2} \bar{u}(p_1, s_1)v(p_2, s_2) \times \bar{v}(p_2, s_2)u(p_2, s_1) \\
&=\; g^2 \operatorname{tr}\left(\left(\sum_{s_1} u(p_1, s_1)\bar{u}(p_1, s_1)\right)\left(\sum_{s_2} v(p_2, s_2)\bar{v}(p_2, s_2)\right)\right) \\
&=\; g^2 \operatorname{tr}\Big((\not{p}_1 + m_f)\,(\not{p}_2 - m_f)\Big) \\
&=\; g^2\Big(\operatorname{tr}(\not{p}_1\,\not{p}_2) \;-\; m_f^2 \operatorname{tr}(1)\Big) \;=\; g^2\Big(4p_1p_2 \;-\; 4m_f^2\Big) \\
&=\; g^2\Big(2(p_1 + p_2 = k)^2 \;-\; 8m_f^2\Big) \;=\; 2g^2(M_s^2 \;-\; 4m_f^2).
\end{aligned} \tag{S.2}$$

Finally, for the two-body decay, the phase-space factor evaluates to

$$\frac{1}{2M_s}\int\frac{d^3\mathbf{p}_1}{(2\pi)^3\,2E_1}\int\frac{d^3\mathbf{p}_2}{(2\pi)^3\,2E_2}\,(2\pi^4)\delta^{(4)}(p_1 + p_2 - k) \;=\; \frac{|\mathbf{p}_1||}{8\pi M_s^2} \tag{S.3}$$

where

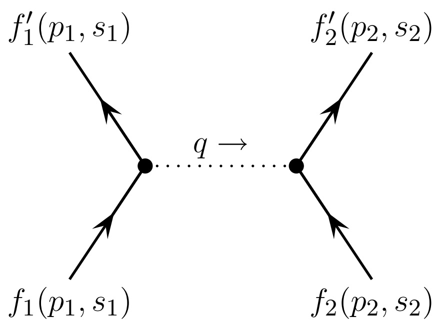
$$|\mathbf{p}_1| \;=\; |\mathbf{p}_2| \;=\; \sqrt{(\tfrac{1}{2}M_s)^2 \;-\; m_f^2}\,. \tag{S.4}$$

Putting all these factors together, we arrive at

$$\Gamma(S \to f + \bar{f}) \;=\; \frac{|\mathbf{p}_1||}{8\pi M_s^2} \times \sum_{s_1,s_2} |\mathcal{M}|^2 \;=\; \frac{g^2 M_s}{8\pi} \times \left(1 - \frac{4m_f^2}{M_s^2}\right)^{3/2}. \tag{S.5}$$

<u>Problem **2**</u>:

As discussed in class, scattering of fermions in the Yukawa theory proceeds by exchange of virtual scalar quanta between the fermions. For the problem at hand, the two fermions are distinct rather than identical, hence only one tree-level Feynman diagram contributes to the scattering:



$$q = p_1 - p'_1 = p'_2 - p_2\,, \tag{S.6}$$

and therefore

$$\mathcal{M}_{\text{tree}}(f_1 + f_2 \to f_1 + f_2) \;=\; -\frac{g_1 g_2}{q^2 - M_s^2} \times \bar{u}(p'_1, s'_1) u(p_1, s_1) \times \bar{u}(p'_2, s'_2) u(p_2, s_2). \tag{S.7}$$

For specific spin states of all particles, the partial cross section of elastic scattering is given by

$$\frac{d\sigma}{d\Omega_{\text{c.m.}}} \;=\; \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{c.m.}}^2}\,, \tag{S.8}$$

but for un-polarized beams and spin-blind detectors we should sum this formula over the final particles' spins and average over spins of the initial particles, thus

$$\begin{aligned} \frac{d\sigma}{d\Omega_{\text{c.m.}}} \;&=\; \frac{1}{64\pi^2 E_{\text{c.m.}}^2} \times \tfrac{1}{2} \sum_{s_1} \tfrac{1}{2} \sum_{s_2} \sum_{s'_1} \sum_{s'_2} |\mathcal{M}|^2 \\ &=\; \frac{1}{256\pi^2 E_{\text{c.m.}}^2} \times \left(\frac{g_1 g_2}{q^2 - M_s^2}\right)^2 \times \sum_{s_1,s'_1} \left|\bar{u}(p'_1, s'_1) u(p_1, s_1)\right|^2 \times \sum_{s_2,s'_2} \left|\bar{u}(p'_2, s'_2) u(p_2, s_2)\right|^2. \end{aligned} \tag{S.9}$$

The spin sums on the second line here are similar to the spin sum in eq. (S.2):

$$\begin{aligned}
\sum_{s_1,s_1'} \left|\bar u(p_1',s_1')u(p_1,s_1)\right|^2 \;&=\; \operatorname{tr}\left(\left(\sum_{s_1} u(p_1,s_1)\bar u(p_1,s_1)\right)\left(\sum_{s_1'} u(p_1',s_1')\bar u(p_1',s_1')\right)\right) \\
&=\; \operatorname{tr}\Big((\not p_1+m_1)\,(\not p_1'+m_1)\Big) \\
&=\; 4p_1p_1' \;+\; 4m_1^2 \;=\; 8m_1^2 \;-\; 2(p_1-p_1')^2 \\
&=\; 8m_1^2 \;-\; 2q^2
\end{aligned} \tag{S.10}$$

and likewise

$$\sum_{s_2,s_2'} \left|\bar u(p_2',s_2')u(p_2,s_2)\right|^2 \;=\; 8m_2^2 \;-\; 2q^2,$$

therefore

$$\frac{d\sigma}{d\Omega_{\rm c.m.}} \;=\; \frac{g_1^2g_2^2}{64\pi^2E^2_{\rm c.m.}}\times\frac{(4m_1^2-q^2)(4m_2^2-q^2)}{(q^2-M_s^2)^2}\,. \tag{S.11}$$

Finally, we should integrate over the scattered particles' directions and calculate the total cross-section. In the center-of-mass frame, $q^0=0$, $\mathbf{q}^2=(\mathbf{p}_1-\mathbf{p}_1')^2=2\mathbf{p}_1^2(1-\cos\theta)$, hence

$$d\Omega \;=\; 2\pi\,d(-\cos\theta) \;=\; \frac{2\pi}{2\mathbf{p}^2}\,d\mathbf{q}^2. \tag{S.12}$$

Consequently, substituting $q^2=-\mathbf{q}^2$ in eq. (S.11) and integrating over $d\mathbf{q}^2$, we arrive at

$$\begin{aligned}
\sigma_{\rm tot} \;&=\; \frac{g_1^2g_2^2}{64\pi^2E^2_{\rm c.m.}}\times\frac{2\pi}{2\mathbf{p}^2}\int\limits_0^{4\mathbf{p}^2} d\mathbf{q}^2\,\frac{(4m_1^2+\mathbf{q}^2)(4m_2^2+\mathbf{q}^2)}{(M_s^2+\mathbf{q}^2)^2} \\
&=\; \frac{g_1^2g_2^2}{16\pi E^2_{\rm c.m.}}\left[1\;+\;\frac{(4m_1^2-M_s^2)(4m_2^2-M_s^2)}{M_s^2(M_s^2+4\mathbf{p}^2)}\;+\;\frac{2m_1^2+2m_2^2-M_s^2}{2\mathbf{p}^2}\log\frac{M_s^2+4\mathbf{p}^2}{M_s^2}\right]
\end{aligned} \tag{S.13}$$

where

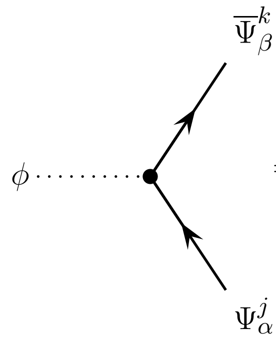
$$\mathbf{p}^2 \;=\; \tfrac14 E^2_{\rm c.m.} \;-\; \tfrac12(m_1^2+m_2^2) \;+\; \frac{(m_1^2-m_2^2)^2}{4E^2_{\rm c.m.}} \tag{S.14}$$

is the solution of the kinematical relation

$$E_{\rm c.m.} \;=\; E_1 \;+\; E_2 \;=\; \sqrt{m_1^2+\mathbf{p}^2} \;+\; \sqrt{m_2^2+\mathbf{p}^2}.$$

Problem **3**(a):

At the free-field level, the intrinsic parities of particles do not matter, and the free Lagrangian of the "pseudo–Yukawa" theory is no different from the "true Yukawa" theory studied in class (*cf.* the top lines of eqs. (1) and (2)). Consequently, the "pseudo–Yukawa" Feynman rules have exactly the same propagators, external-leg factors and the sign rules as studied in class. On the other hand, the interaction terms of the "pseudo–Yukawa" theory are different (*cf.* the bottom lines of eqs. (1) and (2)), and this leads to different vertices in the Feynman rules, namely

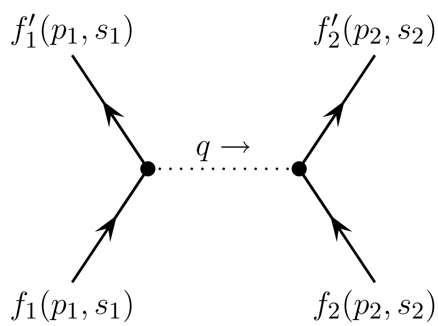


$$= \delta^{jk} g_j \times (\gamma^5)_{\alpha\beta} \qquad \text{(S.15)}$$

where the indices $j, k = 1, 2$ refer to the two species of Dirac fermions.

Problem **3**(b):

In any version of the Yukawa theory, fermions interact with each other via exchanges of scalar (or pseudoscalar) quanta. As in problem (2), for elastic scattering of two distinct fermions there is only one tree level diagram



$$q = p_1 - p'_1 = p'_2 - p_2\,, \qquad \text{(S.16)}$$

hence

$$\mathcal{M}_{\text{tree}}(f_1 + f_2 \to f_1 + f_2) \;=\; +\frac{g_1 g_2}{q^2 - M_p^2} \times \bar{u}(p'_1, s'_1)\gamma^5 u(p_1, s_1) \times \bar{u}(p'_2, s'_2)\gamma^5 u(p_2, s_2). \qquad \text{(S.17)}$$

Note that although the diagram (S.16) looks exactly like (S.6), the amplitude (S.17) is different from (S.17) because of different vertices of the "pseudo–Yukawa" theory.

Problem $\mathbf{3}$(c):

As in problem (2), un-polarized beams of initial fermions and spin-blind detectors for the final fermions mean that we should average $|\mathcal{M}|^2$ over initial spins and sum over the final spins, thus

$$\frac{d\sigma}{d\Omega_{\text{c.m.}}} = \frac{1}{256\pi^2 E_{\text{c.m.}}^2} \times \left(\frac{g_1 g_2}{q^2 - M_p^2}\right)^2 \times \sum_{s_1,s_1'} \left|\bar{u}(p_1',s_1')\gamma^5 u(p_1,s_1)\right|^2 \times \sum_{s_2,s_2'} \left|\bar{u}(p_2',s_2')\gamma^5 u(p_2,s_2)\right|^2 . \tag{S.18}$$

Because of the γ^5 matrices, the spin sums work in a slightly different way:

$$\begin{aligned}
\sum_{s_1,s_1'} \left|\bar{u}(p_1',s_1')\gamma^5 u(p_1,s_1)\right|^2 &= \operatorname{tr}\left(\left(\sum_{s_1} u(p_1,s_1)\bar{u}(p_1,s_1)\right)\gamma^5\left(\sum_{s_1'} u(p_1',s_1')\bar{u}(p_1',s_1')\right)\gamma^5\right) \\
&= \operatorname{tr}\Big((\not{p}_1 + m_1)\gamma^5(\not{p}_1' + m_1)\gamma^5\Big) \\
&\langle\!\langle \text{using } \gamma^5 \not{p} = -\not{p}\gamma^5 \text{ and } \gamma^5\gamma^5 = 1 \rangle\!\rangle \\
&= \operatorname{tr}\Big((\not{p}_1 + m_1)(-\not{p}_1' + m_1)\gamma^5\Big) \\
&= -4p_1p_1' + 4m_1^2 = +2q^2,
\end{aligned} \tag{S.19}$$

and likewise

$$\sum_{s_2,s_2'} \left|\bar{u}(p_2',s_2')\gamma^5 u(p_2,s_2)\right|^2 = +2q^2.$$

Consequently, the partial scattering cross-section in the pseudo–Yukawa theory is

$$\frac{d\sigma}{d\Omega_{\text{c.m.}}} = \frac{g_1^2 g_2^2}{64\pi^2 E_{\text{c.m.}}^2} \times \left(\frac{q^2}{q^2 - M_p^2}\right)^2 . \tag{S.20}$$

Finally, integrating over the directions of the final particles in the center-of-mass frame gives us the total cross section

$$\begin{aligned}
\sigma_{\text{tot}} &= \frac{g_1^2 g_2^2}{64\pi^2 E_{\text{c.m.}}^2} \times \frac{2\pi}{2\mathbf{p}^2} \int\limits_0^{4\mathbf{p}^2} d\mathbf{q}^2 \, \frac{(\mathbf{q}^2)^2}{(M_s^2 + \mathbf{q}^2)^2} \\
&= \frac{g_1^2 g_2^2}{16\pi E_{\text{c.m.}}^2} \left[1 + \frac{M_p^2}{M_p^2 + 4\mathbf{p}^2} - \frac{M_p^2}{2\mathbf{p}^2} \log\frac{M_p^2 + 4\mathbf{p}^2}{M_p^2}\right]
\end{aligned} \tag{S.21}$$

where $\mathbf{p}^2$ is exactly as in eq. (S.14).

Problem **4**:

The complex conjugate of the muon decay amplitude

$$\mathcal{M}(\mu^- \to e^- \nu_\mu \bar{\nu}_e) \;=\; \frac{G_F}{\sqrt{2}} \left[\bar{u}(\nu_\mu)(1-\gamma^5)\gamma^\alpha u(\mu^-)\right] \times \left[\bar{u}(e^-)(1-\gamma^5)\gamma_\alpha v(\bar{\nu}_e)\right] \tag{3}$$

has the opposite sign of the γ^5 because $\bar{\gamma}^5 \equiv \gamma^0(\gamma^5)^\dagger\gamma^0 = -\gamma^5$, thus

$$\mathcal{M}^* \;=\; \frac{G_F}{\sqrt{2}} \left[\bar{u}(\mu^-)\gamma^\beta(1+\gamma^5)u(\nu_\mu)\right] \times \left[\bar{v}(\bar{\nu}_e)\gamma_\beta(1+\gamma^5)u(e^-)\right]. \tag{S.22}$$

Consequently,

$$\begin{aligned} |\mathcal{M}|^2 \;=\; \tfrac{1}{2}G_F^2 & \left[\bar{u}(\nu_\mu)(1-\gamma^5)\gamma^\alpha u(\mu^-)\bar{u}(\mu^-)\gamma^\beta(1+\gamma^5)u(\nu_\mu)\right] \\ & \times \left[\bar{u}(e^-)(1-\gamma^5)\gamma_\alpha v(\bar{\nu}_e)\bar{v}(\bar{\nu}_e)\gamma_\beta(1+\gamma^5)u(e^-)\right] \end{aligned} \tag{S.23}$$

and hence

$$\begin{aligned} \tfrac{1}{2}\sum_{\substack{\text{all}\\ \text{spins}}} |\mathcal{M}|^2 \;&=\; \tfrac{1}{4}G_F^2 \,\mathrm{tr}\Big((1-\gamma^5)\gamma^\alpha(\not{p}_\mu + M_\mu)\gamma^\beta(1+\gamma^5)(\not{p}_{\nu_\mu} + m_{\nu_\mu})\Big) \\ &\qquad \times \mathrm{tr}\Big((1-\gamma^5)\gamma_\alpha(\not{p}_{\bar{\nu}_e} - m_{\nu_e})\gamma_\beta(1+\gamma^5)(\not{p}_e + m_e)\Big) \\ &\approx\; \tfrac{1}{4}G_F^2 \,\mathrm{tr}\Big((1-\gamma^5)\gamma^\alpha(\not{p}_\mu + M_\mu)\gamma^\beta(1+\gamma^5)\,\not{p}_\nu\Big) \\ &\qquad \times \mathrm{tr}\Big((1-\gamma^5)\gamma_\alpha\,\not{p}_{\bar{\nu}}\gamma_\beta(1+\gamma^5)\,\not{p}_e\Big) \end{aligned} \tag{S.24}$$

where the approximation is $m_e \approx 0$; we also make use of $m_{\nu_e} = m_{\nu_\mu} = 0$ (which may be exactly true or just a very good approximation, future data will tell) and simplify notations: $p_\nu \equiv p_{\nu_\mu}$ and $p_{\bar{\nu}} \equiv p_{\bar{\nu}_e}$. Please note that here and henceforth the indices $\mu, \nu, \bar{\nu}, e$ denote the particles to which respective momenta belong and have nothing to do with the Lorentz indices of those momenta. For the Lorentz indices, I use here α, β and later also γ, δ, σ and ρ. Thus, $p_{\mu\alpha}$ is the α's component of the muon's 4–momentum, *etc.*, *etc.*

Having derived eq. (S.24), we now need to evaluate the traces. For the first trace, we have

$$\begin{aligned}
\operatorname{tr}\Big((1-\gamma^5)\gamma^\alpha(\not{p}_\mu+M_\mu)\gamma^\beta(1+\gamma^5)\not{p}_\nu\Big) &= \operatorname{tr}\Big((1-\gamma^5)\gamma^\alpha(\not{p}_\mu+M_\mu)\gamma^\beta\not{p}_\nu(1-\gamma^5)\Big) \\
&= \operatorname{tr}\Big((1-\gamma^5)^2\gamma^\alpha(\not{p}_\mu+M_\mu)\gamma^\beta\not{p}_\nu\Big) \\
&= 2\operatorname{tr}\Big((1-\gamma^5)\gamma^\alpha(\not{p}_\mu+M_\mu)\gamma^\beta\not{p}_\nu\Big) \\
\langle\!\langle \text{using } \operatorname{tr}(\gamma^\alpha\gamma^\beta\not{p}_\nu)=\operatorname{tr}(\gamma^5\gamma^\alpha\gamma^\beta\not{p}_\nu)=0\rangle\!\rangle \qquad & \\
&= 2\operatorname{tr}\Big(\gamma^\alpha\not{p}_\mu\gamma^\beta\not{p}_\nu\Big) - 2\operatorname{tr}\Big(\gamma^5\gamma^\alpha\not{p}_\mu\gamma^\beta\not{p}_\nu\Big) \\
&= 8\Big[p_\mu^\alpha p_\nu^\beta + p_\mu^\beta p_\nu^\alpha - g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big] + 8i\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}.
\end{aligned} \tag{S.25}$$

Similarly, the second trace evaluates to

$$\operatorname{tr}\big((1-\gamma^5)\gamma_\alpha\not{p}_e\gamma_\beta(1+\gamma^5)\not{p}_{\bar\nu}\big) = 8\big[(p_{e\alpha}p_{\bar\nu\beta} + p_{e\beta}p_{\bar\nu\alpha} - g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\big] + 8i\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\,. \tag{S.26}$$

It remains to substitute the trace formulæ (S.25) and (S.26) back into eq. (S.24) and contract the Lorentz indices. Thus,

$$\begin{aligned}
\frac{1}{2}\sum_{\substack{\text{all}\\\text{spins}}}|\mathcal{M}|^2 &= 16G_F^2\left(\Big[p_\mu^\alpha p_\nu^\beta + p_\mu^\beta p_\nu^\alpha - g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big] + i\epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}\right) \\
&\quad\times\left(\Big[p_{e\alpha}p_{\bar\nu\beta} + p_{e\beta}p_{\bar\nu\alpha} - g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\Big] + i\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\right) \\
&\langle\!\langle \text{using symmetry/antisymmetry of factors under } \alpha\leftrightarrow\beta\,\rangle\!\rangle \\
&= 16G_F^2\Bigg(\Big[p_\mu^\alpha p_\nu^\beta + p_\mu^\beta p_\nu^\alpha - g^{\alpha\beta}(p_\mu\cdot p_\nu)\Big]\times\Big[p_{e\alpha}p_{\bar\nu\beta} + p_{e\beta}p_{\bar\nu\alpha} - g_{\alpha\beta}(p_e\cdot p_{\bar\nu})\Big] \\
&\qquad - \epsilon^{\alpha\gamma\beta\delta}p_{\mu\gamma}p_{\nu\delta}\times\epsilon_{\alpha\rho\beta\sigma}p_{\bar\nu}^\rho p_e^\sigma\Bigg) \\
&= 16G_F^2\Bigg(\Big[2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu}) + 2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e) \\
&\qquad - 2(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu}) - 2(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu}) + 4(p_\mu\cdot p_\nu)(p_e\cdot p_{\bar\nu})\Big] \\
&\qquad + \Big[2(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e) - 2(p_\mu\cdot p_e)(p_\nu\cdot p_{\bar\nu})\Big]\Bigg) \\
&= 64G_F^2\,(p_\mu\cdot p_{\bar\nu})(p_\nu\cdot p_e)\,.
\end{aligned} \tag{S.27}$$

$\mathcal{Q.E.D.}$