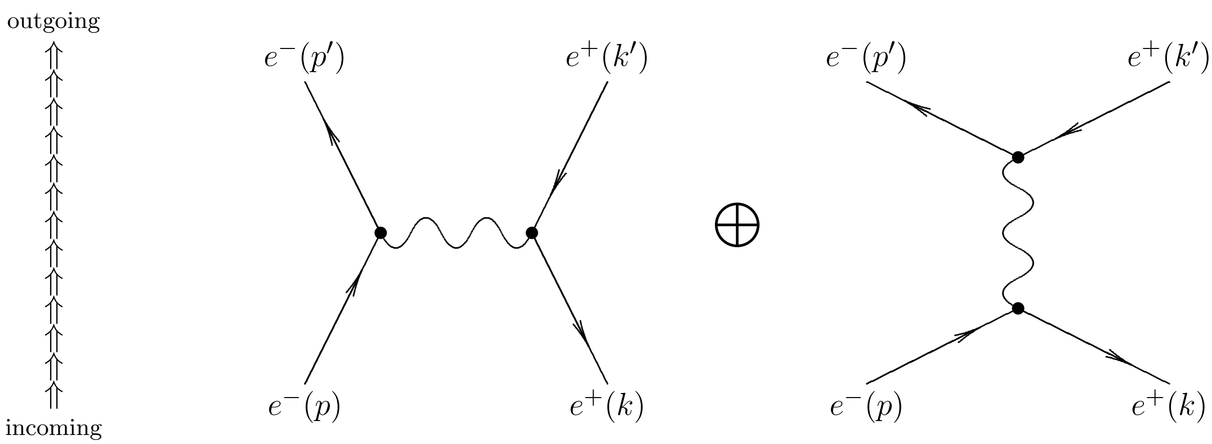
Problem **1**:

At the tree level, Bhabha scattering $e^+e^- \to e^+e^-$ proceeds via the two following Feynman diagrams:



(S.1)

which give us the scattering amplitude

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_1 + \mathcal{M}_2 , \\ \mathcal{M}_1 &= -\frac{e^2}{t} \left[\bar{u}(p')\gamma^\mu u(p)\right] \left[\bar{v}(k)\gamma_\mu v(k')\right], \\ \mathcal{M}_2 &= +\frac{e^2}{s} \left[\bar{v}(k)\gamma^\mu u(p)\right] \left[\bar{u}(p')\gamma_\mu v(k')\right]. \end{aligned} \tag{S.2}$$

Note the relative sign between the two amplitudes: The t–channel diagram has a direct positron-in-positron-out line, hence an overall minus sign; the u–channel diagram does not have this sign factor because all its fermionic lines involve an electron e^- on one end or the other.

Let us sum over the spins; please bear eq. (1) in mind! Starting with the first diagram, we have

$$\begin{aligned} \sum_{\text{spins}} |\mathcal{M}_1|^2 &= \left(\frac{e^2}{t}\right)^2 \sum_{\text{spins}} \left[\bar{u}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu u(p')\right] \left[\bar{v}(k)\gamma_\mu v(k')\bar{v}(k')\gamma_\nu v(k)\right] \\ &= \left(\frac{e^2}{t}\right)^2 \operatorname{tr}\left[(\not{p}' + m)\gamma^\mu(\not{p} + m)\gamma^\nu\right] \operatorname{tr}\left[(\not{k} - m)\gamma_\mu(\not{k}' - m)\gamma_\nu\right] \\ &\approx \left(\frac{e^2}{t}\right)^2 \operatorname{tr}\left[\not{p}'\gamma^\mu\not{p}\gamma^\nu\right] \operatorname{tr}\left[\not{k}\gamma_\mu\not{k}'\gamma_\nu\right] \end{aligned} \tag{S.3}$$

$$
\begin{aligned}
&= \left(\frac{e^2}{t}\right)^2 \times 4\left[p'^{\mu}p^{\nu} + p'^{\nu}p^{\mu} - g^{\mu\nu}(p'\cdot p)\right] \times 4\left[k'_{\mu}k_{\nu} + k'_{\nu}k_{\mu} - g_{\mu\nu}(k'\cdot k)\right] \\
&= 16\left(\frac{e^2}{t}\right)^2 \Big[2(k'\cdot p')(k\cdot p) + 2(k'\cdot p)(k\cdot p') \\
&\qquad - 2(k'\cdot k)(p'\cdot p) - 2(k'\cdot k)(p'\cdot p) + 4(k'\cdot k)(p'\cdot p)\Big] \\
&= 32\left(\frac{e^2}{t}\right)^2 \left[(k'\cdot p')(k\cdot p) + (k'\cdot p)(k\cdot p')\right] \\
&= 8\left(\frac{e^2}{t}\right)^2 \left[s^2 + u^2\right]
\end{aligned}
\tag{S.3}
$$

where the last equality follows from the kinematic relations

$$
\begin{aligned}
2(k\cdot p) &= 2(k'\cdot p') = s - 2m^2 \approx s, \\
2(p\cdot p') &= 2(k\cdot k') = 2m^2 - t \approx -t, \\
2(k\cdot p') &= 2(p\cdot k') = 2m^2 - u \approx -u.
\end{aligned}
\tag{S.4}
$$

In a similar manner, the second diagram yields

$$
\begin{aligned}
\sum_{\text{spins}} |\mathcal{M}_2|^2 &= \left(\frac{e^2}{s}\right)^2 \sum_{\text{spins}} \left[\bar{v}(k)\gamma^{\mu}u(p)\bar{u}(p)\gamma^{\nu}v(k)\right]\left[\bar{u}(p')\gamma_{\mu}v(k')\bar{v}(k')\gamma_{\nu}u(p')\right] \\
&= \left(\frac{e^2}{s}\right)^2 \text{tr}\left[(\not{k} - m)\gamma^{\mu}(\not{p} + m)\gamma^{\nu}\right] \text{tr}\left[(\not{p}' - m)\gamma_{\mu}(\not{k}' - m)\gamma_{\nu}\right] \\
&\approx \left(\frac{e^2}{s}\right)^2 \text{tr}\left[\not{k}\gamma^{\mu}\not{p}\gamma^{\nu}\right] \text{tr}\left[\not{p}'\gamma_{\mu}\not{k}'\gamma_{\nu}\right] \qquad \text{(S.5)} \\
&= \left(\frac{e^2}{s}\right)^2 \times 4\left[k^{\mu}p^{\nu} + k^{\nu}p^{\mu} - g^{\mu\nu}(k\cdot p)\right] \times 4\left[k'_{\mu}p'_{\nu} + k'_{\nu}p'_{\mu} - g_{\mu\nu}(k'\cdot p')\right] \\
&= 16\left(\frac{e^2}{s}\right)^2 \Big[2(k'\cdot k)(p'\cdot p) + 2(k'\cdot p)(k\cdot p'). \\
&\qquad - 2(k'\cdot p')(k\cdot p) - 2(k'\cdot p')(k\cdot p) + 4(k'\cdot p')(k\cdot p)\Big] \\
&= 32\left(\frac{e^2}{s}\right)^2 \left[(k'\cdot k)(p'\cdot p) + (k'\cdot p)(k\cdot p')\right] \\
&= 8\left(\frac{e^2}{s}\right)^2 \left[t^2 + u^2\right].
\end{aligned}
\tag{S.5}
$$

The interference term is more interesting:

$$\begin{aligned}
\sum_{\text{spins}} \mathcal{M}_1^* \mathcal{M}_2 \;&=\; -\frac{e^4}{st} \sum_{\text{spins}} \left[\bar{u}(p)\gamma^\mu u(p')\bar{u}(p')\gamma_\nu v(k')\bar{v}(k')\gamma_\mu v(k)\bar{v}(k)\gamma^\nu u(p)\right] \\
&=\; -\frac{e^4}{st} \times \operatorname{tr}\left[(\not{p}+m)\gamma^\mu(\not{p}'+m)\gamma_\nu(\not{k}'-m)\gamma_\mu(\not{k}-m)\gamma^\nu\right] \\
&\approx\; -\frac{e^4}{st}\, \operatorname{tr}\left[\not{p}\gamma^\mu\not{p}'\gamma_\nu\not{k}'\gamma_\mu\not{k}\gamma^\nu\right] \\
&\langle\!\langle \text{using } \gamma_\nu\not{k}'\gamma_\mu\not{k}\gamma^\nu = -2\not{k}\gamma_\mu\not{k}' \rangle\!\rangle \\
&=\; +2\frac{e^4}{st}\, \operatorname{tr}\left[\not{p}\gamma^\mu\not{p}'\not{k}\gamma_\mu\not{k}'\right] \\
&\langle\!\langle \text{using } \gamma^\mu\not{p}'\not{k}\gamma_\mu = 4(p'\cdot k) \rangle\!\rangle \\
&=\; +8\frac{e^4}{st}\,(p'\cdot k)\, \operatorname{tr}\left[\not{p}\not{k}'\right] \\
&=\; +32\frac{e^4}{st}\,(p'\cdot k)\,(p\cdot k') \\
&=\; +8\frac{e^4}{st}\,u^2.
\end{aligned} \tag{S.6}$$

Substituting the spin sums (S.3), (S.5) and (S.6) into eq. (1) and dividing by 4 (for the 4 spin states of the two initial particles), we arrive at

$$\frac{1}{4}\sum_{\substack{\text{all}\\ \text{spins}}} |\mathcal{M}|^2 \;=\; 2e^4\left[\frac{s^2+u^2}{t^2} \;+\; \frac{t^2+u^2}{s^2} \;+\; \frac{2u^2}{st}\right] \;=\; 2e^4\left[\left(\frac{s}{t}\right)^2 \;+\; \left(\frac{t}{s}\right)^2 \;+\; \left(\frac{u}{s}+\frac{u}{t}\right)^2\right] \tag{S.7}$$

and hence

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} \;=\; \frac{\alpha^2}{2s}\left[\left(\frac{s}{t}\right)^2 \;+\; \left(\frac{t}{s}\right)^2 \;+\; \left(\frac{u}{s}+\frac{u}{t}\right)^2\right] \tag{4}$$

$$\begin{aligned}
&=\; \frac{\alpha^2}{2s} \times \frac{s^4+t^4+u^4}{s^2\,t^2} \\
&=\; \frac{\alpha^2}{4E_{\text{c.m.}}^2} \times \frac{(3+\cos^2\theta)^2}{(1-\cos\theta)^2}\,.
\end{aligned} \tag{S.8}$$

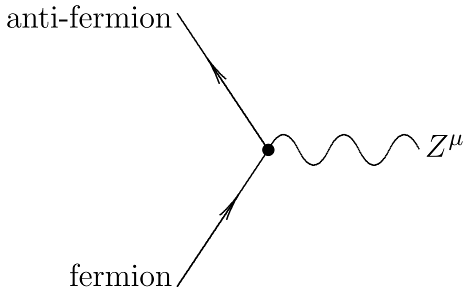
The last two equalities are left as a little extra exercise for the students.

Problem **2**:

The Feynman propagator for the massive vector field Z^μ was derived in homework set #5 to be

$$= \quad \frac{-i}{q^2 - M_Z^2 + i0}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{M_Z^2}\right), \tag{S.9}$$

while the fermion–Z vertices follow from the neutral current (4):



$$= \quad -ig'(g_V + g_A\gamma^5)\gamma^\mu \,. \tag{S.10}$$

Given these Feynman rules, the tree-level decay amplitude of the Z^0 particle into a lepton and anti-lepton or a quark and an anti-quark is simply

$$\mathcal{M}(Z \to f\bar{f}) \;=\; g' e_\mu(Z) \times \bar{u}(f)(g_V + g_A\gamma^5)\gamma^\mu v(\bar{f}) \tag{S.11}$$

where $e_\mu(Z)$ depends on the polarization of the Z particle. In homework set #5, the polarization vectors $e^\mu(k_Z, \lambda_Z)$ were denoted $f^\mu(k, \lambda)$, and they were shown to satisfy

$$k_\mu\, e^\mu(k,\lambda) \;=\; 0, \qquad e^\mu(k,\lambda)e^*_\mu(k,\lambda') \;=\; -\delta_{\lambda,\lambda'} \tag{S.12}$$

and

$$\sum_\lambda e^\mu(k,\lambda)e^{*,\nu}(k,\lambda) \;=\; -g^{\mu\nu} \;+\; \frac{k^\mu k^\nu}{M_Z^2}\,. \tag{S.13}$$

Consequently, averaging the mod-squared decay amplitude (S.11) over the initial Z–particle's spin states gives

$$\begin{aligned}
\tfrac{1}{3}\sum_{\lambda_Z}|\mathcal{M}|^2 \;&=\; \frac{g'^2}{3}\left(-g^{\mu\nu} \;+\; \frac{k_Z^\mu k_Z^\nu}{M_Z^2}\right) \times \Big(\bar{u}(f)(g_V + g_A\gamma^5)\gamma_\mu v(\bar{f})\Big) \times \Big(\bar{v}(\bar{f})\overline{(g_V + g_A\gamma^5)\gamma_\nu}u(f)\Big), \\
&=\; -\frac{g'^2}{3}\Big(\bar{u}(f)(g_V + g_A\gamma^5)\gamma_\mu v(\bar{f})\Big) \times \Big(\bar{v}(\bar{f})(g_V + g_A\gamma^5)\gamma^\mu u(f)\Big) \\
&\quad +\; \frac{g'^2}{3M_Z^2}\left|\bar{v}(\bar{f})(g_V + g_A\gamma^5)\not{k}_Z u(f)\right|^2.
\end{aligned} \tag{S.14}$$

Next, we sum the mod-squared amplitude over the final fermion's and anti-fermion's spin states.

For the second line of eq. (S.14), the spin sum evaluates to

$$\begin{aligned}
&\sum_{s_f,s_{\bar f}} \big(\bar u(f)(g_V+g_A\gamma^5)\gamma_\mu v(\bar f)\big)\times\big(\bar v(\bar f)(g_V+g_A\gamma^5)\gamma^\mu u(f)\big)\\
&\qquad = \operatorname{tr}\Big((\not p_f+m_f)(g_V+g_A\gamma^5)\gamma_\mu(\not p_{\bar f}-m_f)(g_V+g_A\gamma^5)\gamma^\mu\Big)\\
&\qquad = \operatorname{tr}\Big(\not p_f(g_V+g_A\gamma^5)\gamma_\mu\not p_{\bar f}(g_V+g_A\gamma^5)\gamma^\mu\Big) \;-\; m_f^2\operatorname{tr}\big((g_V+g_A\gamma^5)\gamma_\mu(g_V+g_A\gamma^5)\gamma^\mu\big)\\
&\qquad = \operatorname{tr}\Big(\not p_f(g_V+g_A\gamma^5)^2\gamma_\mu\not p_{\bar f}\gamma^\mu\Big) \;-\; m_f^2\operatorname{tr}\big((g_V+g_A\gamma^5)(g_V-g_A\gamma^5)\gamma_\mu\gamma^\mu\big)\\
&\qquad = \operatorname{tr}\Big(\not p_f(g_V^2+g_A^2+2g_Vg_A\gamma^5)(-2\not p_{\bar f})\Big) \;-\; m_f^2\operatorname{tr}\big((g_V^2-g_A^2)(4)\big)\\
&\qquad = -8(g_V^2+g_A^2)(p_fp_{\bar f}) \;-\; 16m_f^2(g_V^2-g_A^2),
\end{aligned}\tag{S.15}$$

while For the third line we first simplify

$$\begin{aligned}
\bar v(\bar f)(g_V+g_A\gamma^5)\not k_Z u(f) &= \bar v(\bar f)(g_V+g_A\gamma^5)(\not p_f+\not p_{\bar f})u(f)\\
&= \bar v(\bar f)(g_V+g_A\gamma^5)\big[\not p_f u(f)\big] \;+\; \big[\bar v(\bar f)\not p_{\bar f}\big](g_V-g_A\gamma^5)u(f)\\
&= m_f\,\bar v(\bar f)(g_V+g_A\gamma^5)u(f) \;-\; m_f\,\bar v(\bar f)(g_V-g_A\gamma^5)u(f)\\
&= 2m_fg_A\,\bar v(\bar f)\gamma^5u(f),
\end{aligned}\tag{S.16}$$

and then sum over the fermionic spins:

$$\begin{aligned}
\sum_{s_f,s_{\bar f}}\Big|(2m_fg_A)\,\bar v(\bar f)\gamma^5u(f)\Big|^2 &= (2m_fg_A)^2\operatorname{tr}\Big((\not p_f+m_f)\gamma^5(\not p_{\bar f}-m_f)\overline{\gamma^5}\Big)\\
&\langle\!\langle\text{using }\overline{\gamma^5}=-\gamma^5\rangle\!\rangle\\
&= (2m_fg_A)^2\operatorname{tr}\Big((\not p_f+m_f)(\not p_{\bar f}+m_f)\Big)^2\\
&= 16m_f^2g_A^2\Big((p_fp_{\bar f})+m_f^2\Big).
\end{aligned}\tag{S.17}$$

Substituting formulæ (S.15) and (S.17) into eq. (S.14), we arrive at

$$\frac13\sum_{\text{spins}}^{\text{all}}\big|\mathcal M(Z\to f\bar f)\big|^2 \;=\; \frac{8g'^2g_V^2}{3}\Big[(p_fp_{\bar f})+2m_f^2\Big] \;+\; \frac{8g'^2g_A^2}{3}\left[(p_fp_{\bar f}) \;-\; 2m_f^2 \;+\; 2m_f^2\,\frac{(p_fp_{\bar f})+m_f^2}{M_Z^2}\right].\tag{S.18}$$

Finally, using

$$2m_f^2 \;+\; 2(p_fp_{\bar f}) = \; p_f^2 \;+\; p_{\bar f}^2 \;+\; 2(p_fp_{\bar f}) \;=\; (p_f+p_{\bar f})^2 \;=\; k_Z^2 \;=\; M_Z^2,$$

we simplify

$$\frac{1}{3}\sum_{\rm spins}^{\rm all}\left|\mathcal{M}(Z\to f\bar{f})\right|^2 \;=\; \frac{4g'^2g_V^2}{3}\Big[M_Z^2\,+\,2m_f^2\Big] \;+\; \frac{4g'^2g_A^2}{3}\Big[M_Z^2\,-\,4m_f^2\Big] \\ =\; \tfrac{4}{3}g'^2(g_V^2+g_A^2)M_Z^2 \;+\; \tfrac{8}{3}g'^2(g_V^2-2g_A^2)m_f^2 \tag{S.19}$$

and therefore

$$\Gamma(Z\to f\bar{f}) \;=\; \frac{|\mathbf{p}_f|}{8\pi M_Z^2}\times\frac{1}{3}\sum_{\rm spins}^{\rm all}\left|\mathcal{M}(Z\to f\bar{f})\right|^2 \\ =\; \frac{g'^2M_Z}{12\pi}\times\left((g_V^2+g_A^2)\;+\;\frac{m_f^2}{M_Z^2}\,(2g_V^2-4g_A^2)\right)\times\sqrt{1\,-\,\frac{4m_f^2}{M_Z^2}}\,. \tag{S.20}$$

Given the fermionic masses of the standard model, all the quarks and leptons that the Z particle can decay to are so much lighter than M_z that we may neglect their masses altogether. Consequently,

$$\Gamma(Z\to f\bar{f}) \;\approx\; \frac{g'^2M_Z}{12\pi}\times(g_V^2+g_A^2) \;=\; \frac{\alpha_{\rm QED}M_Z}{3}\times\frac{g_V^2(f)+g_A^2(f)}{\sin^2\theta_W\cos^2\theta_W}\,, \tag{S.21}$$

and thus

$$\Gamma(Z\to e^-e^+)=\Gamma(Z\to\mu^-\mu^+)=\Gamma(Z\to\tau^-\tau^+) \;=\; \frac{\alpha M_Z}{48}\times\frac{1+(1-4\sin^2\theta_W)^2}{\sin^2\theta_W\cos^2\theta_W} \\ \approx\; 84.4\ \text{MeV}, \tag{S.22}$$

$$\Gamma(Z\to\nu_e\bar{\nu}_e)=\Gamma(Z\to\nu_\mu\bar{\nu}_\mu)=\Gamma(Z\to\nu_\tau\bar{\nu}_\tau) \;=\; \frac{\alpha M_Z}{48}\times\frac{2}{\sin^2\theta_W\cos^2\theta_W} \\ \approx\; 168\ \text{MeV}, \tag{S.23}$$

$$\Gamma(Z\to d\bar{d}) \;=\Gamma(Z\to s\bar{s}) \;=\Gamma(Z\to b\bar{b}) \;=\; 3\times\frac{\alpha M_Z}{48}\times\frac{1+(1-\frac{4}{3}\sin^2\theta_W)^2}{\sin^2\theta_W\cos^2\theta_W} \\ \approx\; 372\ \text{MeV}, \tag{S.24}$$

$$\Gamma(Z\to u\bar{u}) \;=\Gamma(Z\to c\bar{c}) \;=\; 3\times\frac{\alpha M_Z}{48}\times\frac{1+(1+\frac{8}{3}\sin^2\theta_W)^2}{\sin^2\theta_W\cos^2\theta_W} \\ \approx\; 288\ \text{MeV}, \tag{S.25}$$

where the quark-antiquark decay modes gain an additional factor of 3 because the quarks come in 3 colors. Hence,

$$\Gamma\bigl(Z\to\text{any}\,q\bar q\to\text{hadrons}\bigr)\;=\;\frac{\alpha M_Z}{16}\times\frac{10\;-\;\frac{56}{3}\sin^2\theta_W\;+\;\frac{176}{9}\sin^4\theta_W}{\sin^2\theta_W\cos^2\theta_W}\;\approx\;1,693\ \text{MeV}\qquad\text{(S.26)}$$

and the net decay rate (measured as the Z–resonance width)

$$\Gamma_{\text{tot}}(Z)\;=\;\frac{\alpha M_Z}{16}\times\frac{14\;-\;\frac{80}{3}\sin^2\theta_W\;+\;\frac{320}{9}\sin^4\theta_W}{\sin^2\theta_W\cos^2\theta_W}\;\approx\;2,450\ \text{MeV}\qquad\text{(S.27)}$$

The branching ratios follow from eqs. (S.22), (S.23) (S.26) and (S.27):

$$B(Z\to e^-e^+)=B(Z\to\mu^-\mu^+)=B(Z\to\tau^-\tau^+)\;=\;\frac{3\;-\;12\sin^2\theta_W\;+\;24\sin^4\theta_W}{63\;-\;120\sin^2\theta_W\;+\;160\sin^4\theta_W}\;\approx\;3.45\%,$$

$$B(Z\to\text{any}\,\nu\bar\nu)\;=\;\frac{9}{63\;-\;120\sin^2\theta_W\;+\;160\sin^4\theta_W}\;\approx\;20\%,$$

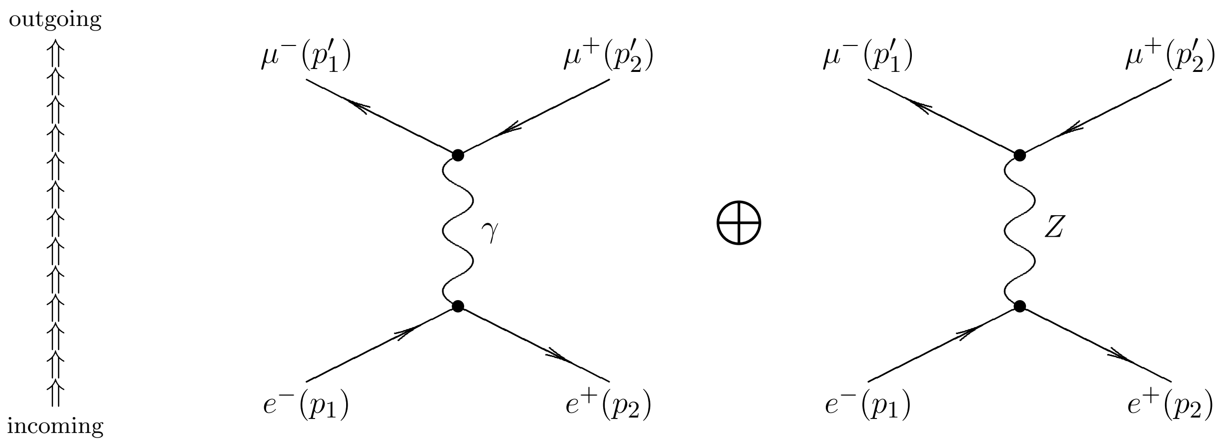
$$B(Z\to\text{hadrons})\;=\;\frac{45\;-\;84\sin^2\theta_W\;+\;88\sin^4\theta_W}{63\;-\;120\sin^2\theta_W\;+\;160\sin^4\theta_W}\;\approx\;70\%.$$

Note 1: The numerical values of the decay rates in eqs. (S.22) through (S.25) and hence (S.26) and (S.27) are based upon $\alpha_{\text{QED}}^{\text{eff}}(M_Z)\approx 1/128$ rather than $\alpha_{\text{QED}}(0)\approx 1/137$. This is a renormalization effect we shall discuss later in the second half of the QFT course. Fortunately, this correction cancels out of the branching ratios.

Note 2: Experimentally, the partial Z decay rates into leptons are within one percent of the rates computed in eq. (S.22) and (S.23). For the decay into hadrons, the experimental decay rate is slightly higher: $\Gamma_{\text{exp}}(Z\to\text{hadrons})\approx 1,745$ MeV, *cf.* eq. (S.26). The discrepancy is due to strong interactions.

Problem **3**(a):

At the tree level of the Standard Model, the $e^-e^+ \to \mu^-\mu^+$ pair production proceeds via two Feynman diagrams, one with the s–channel virtual photon, and the other with the s–channel virtual Z^0:



(S.28)

which together give us the pair-production amplitude

$$\mathcal{M}e^-e^+ \to \mu^-\mu^+ \;=\; \mathcal{M}_\gamma \;+\; \mathcal{M}_Z\,, \tag{S.29}$$

$$\mathcal{M}_\gamma \;=\; \frac{e^2}{s} \times \left[\bar{u}(\mu^-)\gamma_\alpha v(\mu^+)\right] \times \left[\bar{v}(e^+)\gamma^\alpha u(e^-)\right], \tag{S.30}$$

$$\mathcal{M}_Z \;=\; \frac{g'^2}{s - M_Z^2}\left(g_{\alpha\beta} \;-\; \frac{q_Z^\alpha q_Z^\beta}{M_Z^2}\right) \times \left[\bar{u}(\mu^-)(g_V + g_A\gamma^5)\gamma_\alpha v(\mu^+)\right] \times \left[\bar{v}(e^+)(g_V + g_A\gamma^5)\gamma_\beta u(e^-)\right] \tag{S.31}$$

(the electron and the muon have the same $g_V = \sin^2\theta_W - \frac{1}{4} \approx 0.02$ and $g_A = +\frac{1}{4}$, *cf.* eq.(5)).

Thanks to eq. (S.16), we can simplify the second amplitude here to

$$\mathcal{M}_Z \;=\; \frac{g'^2}{s - M_Z^2}\Bigg\{ \left[\bar{u}(\mu^-)(g_V + g_A\gamma^5)\gamma_\alpha v(\mu^+)\right] \times \left[\bar{v}(e^+)(g_V + g_A\gamma^5)\gamma^\alpha u(e^-)\right] \;-\; \frac{4m_e m_\mu g_A^2}{M_Z^2}\left[\bar{u}(\mu^-)\gamma^5 v(\mu^+)\right] \times \left[\bar{v}(e^+)\gamma^5 u(e^-)\right]\Bigg\}, \tag{S.32}$$

where the second term is negligibly small for $m_e \ll M_Z$ and $m_\mu \ll M_Z$. In fact, for any pair-production process $e^-e^+ \to f\bar{f}$, the second term in the amplitude (S.32) is negligibly small thanks to $m_e \ll M_Z$, regardless of whether the outgoing fermions are lighter or heavier than the Z boson.

PS: Actually, in the complete Standard model there are *three* tree-level diagrams contributing to the pair-production process: The two diagrams depicted on fig. (S.28), plus another diagram with the Higgs boson in the s channel, hence

$$\mathcal{M}(e^-e^+ \to \mu^-\mu^+) \;=\; \mathcal{M}_\gamma \;+\; \mathcal{M}_Z \;+\; \mathcal{M}_H \tag{S.33}$$

where

$$\mathcal{M}_{\mathcal{H}} \;=\; \frac{g_{eH}\,g_{\mu H}}{S - M_H^2} \times \left[\bar{u}(\mu^-)v(\mu^+)\right] \times \left[\bar{v}(e^+)u(e^-)\right]. \tag{S.34}$$

The reason we neglect the Higgs's contribution in this exercise is that the Higgs-electron coupling g_{eH} is very small and even the muon–Higgs coupling is quite small. In general, the fermion–Higgs couplings of the Standard Model are proportional to the fermion's masses,

$$g_{eH} \;\equiv\; \frac{\lambda_e}{\sqrt{2}} \;=\; \frac{m_e}{\sqrt{2}\,\langle H\rangle}\,, \qquad g_{\mu H} \;\equiv\; \frac{\lambda_\mu}{\sqrt{2}} \;=\; \frac{m_\mu}{\sqrt{2}\,\langle H\rangle}\,, \qquad \textit{etc.} \tag{S.35}$$

where $\langle H\rangle$ is the Higgs field's expectation value. Likewise,

$$M_Z \;=\; \tfrac{1}{2}g'\,\langle H\rangle \tag{S.36}$$

and therefore $\langle H\rangle \approx 245$ GeV. Consequently, $g_{eH} \approx 1.45\cdot 10^{-6} \ll 1$ and $g_{\mu H} \approx 3.1\cdot 10^{-4} \ll 1$, — and that's why the Higgs' contribution (S.34) to the pair production is indeed extremely small.

Problem **3**(b):

The ultra-relativistic limit of the amplitude $\mathcal{M}_\gamma$ was discussed in class in detail, so let me simply summarize the relevant issues. First, because the matrix

$$\gamma^0\gamma^\alpha \;=\; \begin{pmatrix} \bar\sigma^\alpha & 0 \\ 0 & \sigma^\alpha \end{pmatrix} \tag{S.37}$$

is block-diagonal in the Weyl basis, the incoming electron and positron must have similar chiralities and hence opposite helicities; otherwise,

$$\bar{v}(e^+_L)\gamma^\alpha u(e^-_L) \;=\; \bar{v}(e^+_R)\gamma^\alpha u(e^-_R) \;=\; 0. \tag{S.38}$$

Furthermore, in the center of mass frame,

$$\begin{aligned} \bar{v}(e^+_R)\gamma^\alpha u(e^-_L) \;&=\; 2E(0,+1,-i,0)^\alpha, \\ \bar{v}(e^+_L)\gamma^\alpha u(e^-_R) \;&=\; 2E(0,-1,-i,0)^\alpha. \end{aligned} \tag{S.39}$$

Likewise, for the outgoing muons we have

$$\begin{aligned}
\bar{u}(\mu_L^-)\gamma^\beta v(\mu_L^+) &= 0, \\
\bar{u}(\mu_R^-)\gamma^\beta v(\mu_R^+) &= 0, \\
\bar{u}(\mu_L^-)\gamma^\beta v(\mu_R^+) &= 2E(0, +\cos\theta, +i, +\sin\theta)^\beta, \\
\bar{u}(\mu_R^-)\gamma^\beta v(\mu_L^+) &= 2E(0, -\cos\theta, +i, -\sin\theta)^\beta.
\end{aligned} \tag{S.40}$$

and consequently,

$$\begin{aligned}
\mathcal{M}_\gamma(e_L^- e_R^+ \to \mu_L^- \mu_R^+) &= -e^2(1+\cos\theta), \\
\mathcal{M}_\gamma(e_L^- e_R^+ \to \mu_R^- \mu_L^+) &= -e^2(1-\cos\theta), \\
\mathcal{M}_\gamma(e_R^- e_L^+ \to \mu_L^- \mu_R^+) &= -e^2(1-\cos\theta), \\
\mathcal{M}_\gamma(e_R^- e_L^+ \to \mu_R^- \mu_L^+) &= -e^2(1+\cos\theta),
\end{aligned} \tag{S.41}$$

while for all other helicity combinations, the $\mathcal{M}_\gamma$ amplitudes vanish (in the ultra-relativistic limit).

Now consider the $\mathcal{M}_Z$ amplitudes. In the ultra-relativistic limit, we neglect the second term on the right hand side of eq. (S.32), thus

$$\mathcal{M}_Z \approx \frac{g'^2}{s - M_Z^2} \times \left[\bar{u}(\mu^-)(g_V + g_A\gamma^5)\gamma_\alpha v(\mu^+)\right] \times \left[\bar{v}(e^+)(g_V + g_A\gamma^5)\gamma^\alpha u(e^-)\right], \tag{S.42}$$

which has a similar form to eq. (S.30) for the $\mathcal{M}_\gamma$ amplitude, except for the overall coefficient

$$\frac{g'^2}{s - M_Z^2} \qquad \text{versus} \qquad \frac{e^2}{s}$$

and the parity-violating vertex factors

$$(g_V + g_A\gamma^5)\gamma^\alpha \qquad \text{versus} \qquad \gamma^\alpha.$$

Nevertheless, similar to eq. (S.37), the matrices

$$\gamma^0(g_V + g_A\gamma^5)\gamma^\alpha = \begin{pmatrix} g_L\bar{\sigma}^\alpha & 0 \\ 0 & g_R\sigma^\alpha \end{pmatrix} \tag{S.43}$$

are also block-diagonal in the Weyl basis, with similar consequences: The incoming electron and positron must have opposite helicities, otherwise

$$\bar{v}(e^+_L)(g_V + g_A\gamma^5)\gamma^\alpha u(e^-_L) \;=\; \bar{v}(e^+_R)(g_V + g_A\gamma^5)\gamma^\alpha u(e^-_R) \;=\; 0, \tag{S.44}$$

and likewise, the outgoing muons must have opposite helicities, otherwise

$$\bar{u}(\mu^-_L)(g_V + g_A\gamma^5)\gamma^\alpha v(\mu^+_L)\ \bar{u}(\mu^-_R)(g_V + g_A\gamma^5)\gamma^\alpha v(\mu^+_R) \;=\; 0. \tag{S.45}$$

Furthermore, similar to eqs. (S.39) and (S.40) derived in class, in the center-of-mass frame

$$\begin{aligned} \bar{v}(e^+_R)(g_V + g_A\gamma^5)\gamma^\alpha u(e^-_L) &= 2Eg_L(0,+1,-i,0)^\alpha, \\ \bar{v}(e^+_L)(g_V + g_A\gamma^5)\gamma^\alpha u(e^-_R) &= 2Eg_R(0,-1,-i,0)^\alpha, \end{aligned} \tag{S.46}$$

and likewise,

$$\begin{aligned} \bar{u}(\mu^-_L)(g_V + g_A\gamma^5)\gamma^\beta v(\mu^+_R) &= 2Eg_L(0,+\cos\theta,+i,+\sin\theta)^\beta, \\ \bar{u}(\mu^-_R)(g_V + g_A\gamma^5)\gamma^\beta v(\mu^+_L) &= 2Eg_R(0,-\cos\theta,+i,-\sin\theta)^\beta. \end{aligned} \tag{S.47}$$

Consequently,

$$\begin{aligned} \mathcal{M}_Z(e^-_L e^+_R \to \mu^-_L \mu^+_R) &= -\frac{g'^2 s}{s - M_Z^2}\, g_L^2\,(1+\cos\theta), \\ \mathcal{M}_Z(e^-_L e^+_R \to \mu^-_R \mu^+_L) &= -\frac{g'^2 s}{s - M_Z^2}\, g_L g_R\,(1-\cos\theta), \\ \mathcal{M}_Z(e^-_R e^+_L \to \mu^-_L \mu^+_R) &= -\frac{g'^2 s}{s - M_Z^2}\, g_L g_R\,(1-\cos\theta), \\ \mathcal{M}_Z(e^-_R e^+_L \to \mu^-_R \mu^+_L) &= -\frac{g'^2 s}{s - M_Z^2}\, g_R^2\,(1+\cos\theta), \end{aligned} \tag{S.48}$$

while for all other helicity combinations, the $\mathcal{M}_Z$ amplitudes vanish (in the ultra-relativistic limit).

It remains to combine the photon-mediated amplitudes (S.41) with the Z–mediated amplitudes (S.48). Note that both amplitudes have exactly the same helicity structures and the same angular dependence, the only differences being the energy dependence and the overall coefficients. Thus, there are four non-zero definite-helicity pair-production amplitudes in the Standard model, namely

$$
\begin{aligned}
\mathcal{M}(e^-_L e^+_R \to \mu^-_L \mu^+_R) &= -\left(e^2 + g'^2 g_L^2 \frac{s}{s - M_Z^2}\right) \times (1+\cos\theta),\\
\mathcal{M}(e^-_L e^+_R \to \mu^-_R \mu^+_L) &= -\left(e^2 + g'^2 g_L g_R \frac{s}{s - M_Z^2}\right) \times (1-\cos\theta),\\
\mathcal{M}(e^-_R e^+_L \to \mu^-_L \mu^+_R) &= -\left(e^2 + g'^2 g_L g_R \frac{s}{s - M_Z^2}\right) \times (1-\cos\theta),\\
\mathcal{M}(e^-_R e^+_L \to \mu^-_R \mu^+_L) &= -\left(e^2 + g'^2 g_R^2 \frac{s}{s - M_Z^2}\right) \times (1+\cos\theta).
\end{aligned}
\tag{S.49}
$$

Problem **3**(c):

We begin with partial cross-sections for definite helicities. In light of eqs. (S.49), we have

$$
\begin{aligned}
\frac{d\sigma(e^-_L e^+_R \to \mu^-_L \mu^+_R)}{d\Omega_{\text{c.m.}}} &= \frac{\alpha^2_{\text{QED}}}{4s}\left(1 + \frac{(x-\frac{1}{2})^2}{x(1-x)} \cdot \frac{s}{s-M_Z^2}\right)^2 \times (1+\cos\theta)^2,\\
\frac{d\sigma(e^-_L e^+_R \to \mu^-_R \mu^+_L)}{d\Omega_{\text{c.m.}}} &= \frac{\alpha^2_{\text{QED}}}{4s}\left(1 + \frac{x(x-\frac{1}{2})}{x(1-x)} \cdot \frac{s}{s-M_Z^2}\right)^2 \times (1-\cos\theta)^2,\\
\frac{d\sigma(e^-_R e^+_L \to \mu^-_L \mu^+_R)}{d\Omega_{\text{c.m.}}} &= \frac{\alpha^2_{\text{QED}}}{4s}\left(1 + \frac{x(x-\frac{1}{2})}{x(1-x)} \cdot \frac{s}{s-M_Z^2}\right)^2 \times (1-\cos\theta)^2,\\
\frac{d\sigma(e^-_R e^+_L \to \mu^-_R \mu^+_L)}{d\Omega_{\text{c.m.}}} &= \frac{\alpha^2_{\text{QED}}}{4s}\left(1 + \frac{x^2}{x(1-x)} \cdot \frac{s}{s-M_Z^2}\right)^2 \times (1+\cos\theta)^2,
\end{aligned}
\tag{S.50}
$$

where

$$
s = E^2_{\text{c.m.}}, \qquad x \stackrel{\text{def}}{=} \sin^2\theta_W \approx 0.23,
\tag{S.51}
$$

and we made explicit use of

$$
g_L = x - \tfrac{1}{2}, \qquad g_R = x, \qquad g' = \frac{e}{\sin\theta_W \cos\theta_W} \Rightarrow g'^2 = \frac{e^2}{x(1-x)}\,.
\tag{S.52}
$$

For the sake of calculational simplicity, we approximate $x \approx \frac{1}{4}$, thus

$$
\frac{(x-\frac{1}{2})^2}{x(1-x)} \approx \frac{x^2}{x(1-x)} \approx +\frac{1}{3} \qquad \frac{x(x-\frac{1}{2})}{x(1-x)} \approx -\frac{1}{3}\,,
$$

and therefore

$$\begin{aligned}
\frac{d\sigma(e^-_L e^+_R \to \mu^-_L \mu^+_R)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_R e^+_L \to \mu^-_R \mu^+_L)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2_{\text{QED}}}{4s}\left(\frac{4s-3M_Z^2}{3(s-M_Z^2)}\right)^2 \times (1+\cos\theta)^2, \\
\frac{d\sigma(e^-_L e^+_R \to \mu^-_R \mu^+_L)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_R e^+_L \to \mu^-_L \mu^+_R)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2_{\text{QED}}}{4s}\left(\frac{2s-3M_Z^2}{3(s-M_Z^2)}\right)^2 \times (1-\cos\theta)^2,
\end{aligned} \tag{S.53}$$

Next, we integrate over the directions of the outgoing muons. We need to consider both the total cross-sections and the asymmetries (6) for two types of angular dependence. For $\frac{d\sigma}{d\Omega} \propto (1+\cos\theta)^2$ we have

$$\begin{aligned}
\int\limits_{\theta<\frac{\pi}{2}} d^2\Omega\,(1+\cos\theta)^2 &= 2\pi\int\limits_0^{+1} d\cos\theta\,(1+\cos\theta)^2 = \frac{14\pi}{3}\,, \\
\int\limits_{\theta>\frac{\pi}{2}} d^2\Omega\,(1+\cos\theta)^2 &= 2\pi\int\limits_{-1}^{0} d\cos\theta\,(1+\cos\theta)^2 = \frac{2\pi}{3}\,,
\end{aligned} \tag{S.54}$$

and hence

$$\sigma_{\text{tot}} \propto \frac{16\pi}{3} \qquad \text{and} \quad A = +\frac{3}{4}\,. \tag{S.55}$$

For the other angular dependence, $\frac{d\sigma}{d\Omega} \propto (1-\cos\theta)^2$, we use $\theta \to \pi - \theta$ symmetry and immediately obtain

$$\sigma_{\text{tot}} \propto \frac{16\pi}{3} \qquad \text{and} \quad A = -\frac{3}{4}\,. \tag{S.56}$$

Consequently, for the four allowed helicity combinations, we have

$$\begin{aligned}
A &= +\frac{3}{4}\,, \qquad \sigma_{\text{tot}} = \frac{4\pi\alpha^2}{3s}\left(\frac{4s-3M_Z^2}{3(s-M_Z^2)}\right)^2 \qquad \text{for } e^-_L e^+_R \to \mu^-_L \mu^+_R \text{ or } e^-_R e^+_L \to \mu^-_R \mu^+_L\,, \\
A &= -\frac{3}{4}\,, \qquad \sigma_{\text{tot}} = \frac{4\pi\alpha^2}{3s}\left(\frac{2s-3M_Z^2}{3(s-M_Z^2)}\right)^2 \qquad \text{for } e^-_L e^+_R \to \mu^-_R \mu^+_L \text{ or } e^-_R e^+_L \to \mu^-_L \mu^+_R\,.
\end{aligned} \tag{S.57}$$

It remains to sum over the muons' helicities and average over the electron's and positron's helicities. The net result is total cross-section

$$\sigma_{\rm tot}(e^-e^+\to\mu^-\mu^+)\ =\ \frac{4\pi\alpha^2}{3s}\times\left(1\,+\,\frac{s^2}{9(s-M_Z^2)^2}\right) \tag{S.58}$$

and the asymmetry

$$A\ =\ \frac{\frac34(4s-3M_Z^2)^2\ -\ \frac34(2s-3M_Z^2)^2}{(4s-3M_Z^2)^2\,+\,(2s-3M_Z^2)^2}\ =\ \frac{9s(s-M_Z^2)}{2s^2\,+\,18(s-M_Z^2)^2}\,. \tag{S.59}$$

Note the asymmetry is positive for $s<M_Z^2$ and negative for $s>M_Z^2$.

For your information, without the $x=\frac14$ approximation, one has

$$\begin{aligned}\sigma_{\rm tot}(e^-e^+\to\mu^-\mu^+)\ &=\ \frac{4\pi\alpha^2}{3s}\times\frac{1+2n+(r+n)^2}{r^2}\,,\\ A\ &=\ \frac34\times\frac{2(r+2n)}{1+2n+(r+n)^2}\,,\end{aligned}$$

where I have denoted

$$n\ \stackrel{\rm def}{=}\ (1-4x)^2\ \approx\ 0.0052\qquad\text{and}\quad r\ \stackrel{\rm def}{=}\ 16x(1-x)\,\frac{s-M_Z^2}{s}\ \underset{x=\frac14}{\longrightarrow}\ 3\,\frac{s-M_Z^2}{s}\,.$$

Note that n is quite small, so eqs. (S.58) and (S.59) provide a very good approximation. In fact, the corrections due to $x\neq\frac14$ are smaller than the leading loop corrections in the Standard Model, so we were quite justified in approximating $x\approx\frac14$ at the tree level of analysis.