1. First, consider a theory of a real scalar field $\Phi_1(x)$, a real pseudoscalar $\Phi_2(x)$, and a Dirac spinor $\Psi(x)$ governed by Lagrangian

$$\begin{aligned}\mathcal{L} \;=\; &\overline{\Psi}(i\partial\!\!\!/ - M)\Psi \;+\; \tfrac{1}{2}(\partial_\mu\Phi_1)^2 \;+\; \tfrac{1}{2}(\partial_\mu\Phi_2)^2 \;-\; \tfrac{1}{2}m^2(\Phi_1^2+\Phi_2^2)\\ &-\; g\,\overline{\Psi}(\Phi_1 + i\gamma^5\Phi_2)\Psi \;-\; \tfrac{1}{4}\lambda(\Phi_1^2+\Phi_2^2)^2.\end{aligned} \tag{1}$$

(a) Write down the Feynman rules of this theory.

(b) Assume $m > 2M$ and calculate (to the lowest non-trivial order of the perturbation theory) the decay rates of the scalar and the pseudoscalar particles into fermion + antifermion pairs.

(c) Generally, $\Gamma_S \neq \Gamma_P$, but the two decay rates become equal in the limit of negligible fermionic mass M. This equality follows from the axial symmetry of the Lagrangian (1) for $M = 0$. Write down the explicit action of this symmetry on the fermionic and bosonic fields of the theory and explain why it leads to $\Gamma_S = \Gamma_P$.

(d) Finally, consider the *second*-lowest (*i.e.*, first sub-leading) order of the perturbation theory. Draw Feynman diagrams contributing to the decay amplitudes $\mathcal{M}(S \to f + \bar{f})$ and $\mathcal{M}(P \to f + \bar{f})$ at this order. Apply the Feynman rules but skip the actual calculations.

2. And now consider a different QFT where heavy (*i.e.*, $M_s \gg m_e$) neutral scalar particles have Yukawa–like coupling to electrons, which in turn couple to photons according to the usual QED rules, thus

$$\mathcal{L} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \overline{\Psi}(iD\!\!\!\!/ - m_e)\Psi + \left[\tfrac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi \;-\; \tfrac{1}{2}M_s^2\varphi^2\right] \;+\; g\overline{\Psi}\varphi\Psi. \tag{2}$$

In this theory, an electron and a positron colliding with energy $E_{\text{c.m.}} > M_s$ may annihilate into one photon and one scalar particle, $e^- + e^+ \to \gamma + S$.

(a) Draw tree diagrams for the $e^- + e^+ \to \gamma + S$ process and write down the tree-level matrix element $\langle \gamma S|\,\mathcal{M}\,|e^-e^+\rangle$.

(b) Sum $|\mathcal{M}|^2$ over the photon's polarizations, average over the fermion's spins, and calculate the partial cross-section

$$\frac{d\sigma(e^-e^+ \to \gamma S)}{d\Omega_{\text{c.m.}}}\,.$$

For simplicity, neglect the electron's mass.