PHY–396 K. Mid-term Exam. Due Tuesday, November 2, 2004.

Please do not waste time and paper by copying the posted homework solutions. If you need to use any homework result, simply reference the appropriate question or equation and go ahead. Ditto for anything explicitly derived in class.

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1. In three spacetime dimensions (two space plus one time) an antisymmetric Lorentz tensor $F^{\mu\nu} = -F^{\nu\mu}$ is equivalent to an axial Lorentz vector, $F^{\mu\nu} = \epsilon^{\mu\nu\lambda}F_\lambda$. Consequently, in 3D one can have a massive photon despite unbroken gauge invariance of the electromagnetic field $A_\mu(x)$. Indeed, consider the following Lagrangian:

$$\mathcal{L} \;=\; -\frac{1}{2}\,F_\lambda F^\lambda \;+\; \frac{m}{2}\,F_\lambda A^\lambda \tag{1}$$

where

$$F^\lambda(x) \;=\; \tfrac{1}{2}\epsilon^{\lambda\mu\nu}F_{\mu\nu}(x) \;=\; \epsilon^{\lambda\mu\nu}\partial_\mu A_\nu(x). \tag{2}$$

(a) Show that the action $S = \int d^3x\mathcal{L}$ is gauge invariant (although the Lagrangian (1) is not invariant).

(b) Write down equations of motion for the $A_\mu(x)$ fields and show that they imply Klein–Gordon equations $(\partial^2 + m^2)F_\lambda(x) = 0$.

(c) Write down Noether stress-energy tensor for the theory in question, then add a suitable $\epsilon^{\mu\kappa\lambda}\partial_\kappa\mathcal{K}^\nu_\lambda$ term to make $T^{\mu\nu}$ symmetric and gauge invariant.

Hints: (1) See homework set #1 for a 4D example of a $\mathcal{K}$ correction to the $T^{\mu\nu}_{\text{Noether}}$. (2) Your result should have a simple form in terms of F_λ.

(d) In the quantum theory, the $\hat{A}_\mu(x)$ fields create and annihilate massive photons. Without going through gory details of the canonical quantization procedure, show that despite being massive, the photons have only one polarization (*i.e.*, spin state).

(e) Now let us couple the massive $A_\mu(x)$ field to a charged scalar field $\Phi(x)$. Write down a Lagrangian for the whole system of interacting fields A_μ, Φ, and Φ^*.

(f) What are discrete symmetries of the interacting fields?

(g) Finally, suppose the scalar potential $V(\Phi,\Phi^*)$ has a minimum at $\Phi\neq 0$. What happens to the particle spectrum of this theory? (Note that because of the $\frac{m}{2}F_\lambda A^\lambda$ term, the spectrum is more complicated than in the $d=4$ case studied in class.)

2. Now let's go back to $d=3+1$ dimensions and consider a theory of N massless Dirac spinor fields $\Psi_1(x),\ldots,\Psi_N(x)$ coupled to one massless EM field $A_\mu(x)$. All fermions have the same charge $q=-e$, hence

$$\mathcal{L} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \sum_{i=1}^{N}\overline{\Psi}_i(i\,\partial\!\!\!/ \;+\; eA\!\!\!/)\Psi_i\,. \tag{3}$$

This theory has chiral symmetry $U(N)_L\times U(N)_R$ which acts on the Dirac fields according to

$$\Psi_i'(x) \;=\; \sum_j\left(\frac{1-\gamma^5}{2}\,U^L_{i,j} \;+\; \frac{1+\gamma^5}{2}\,U^R_{i,j}\right)\Psi_j(x) \tag{4}$$

where $U^L_{i,j}$ and $U^R_{i,j}$ are two independent $N\times N$ unitary matrices.

(a) Verify that the Lagrangian (3) is invariant under the chiral symmetry (4), show that the Noether currents of this symmetry are linear combinations of vector and axial currents

$$V^\mu_{i,j} \;=\; \overline{\Psi}_i\gamma^\mu\Psi_j \qquad\text{and}\qquad A^\mu_{i,j} \;=\; \overline{\Psi}_i\gamma^\mu\gamma^5\Psi_j\,, \tag{5}$$

and verify that all these currents are indeed conserved (in the classical theory).

(b) In the quantum theory, expand the net charge operators

$$\hat{Q}^V_{i,j} \;=\int d^3\mathbf{x}\,\hat{V}^0_{i,j} \qquad\text{and}\qquad \hat{Q}^A_{i,j} \;=\int d^3\mathbf{x}\,\hat{A}^0_{i,j} \tag{6}$$

in terms of fermionic creation and annihilation operators. For simplicity, use the helicity basis for particles' spin states, and make full use of $m=0$.

(c) Verify that the charges (6) satisfy the commutation relations of the $U(N)\times U(N)$ generators, namely

$$\begin{aligned}
\left[\hat{Q}^V_{i,j},\hat{Q}^V_{k,\ell}\right] \;&=\; \delta_{j,k}\hat{Q}^V_{i,\ell} \;-\; \delta_{i,\ell}\hat{Q}^V_{k,j}\,,\\
\left[\hat{Q}^V_{i,j},\hat{Q}^A_{k,\ell}\right] \;&=\; \delta_{j,k}\hat{Q}^A_{i,\ell} \;-\; \delta_{i,\ell}\hat{Q}^A_{k,j}\,,\\
\left[\hat{Q}^A_{i,j},\hat{Q}^A_{k,\ell}\right] \;&=\; \delta_{j,k}\hat{Q}^V_{i,\ell} \;-\; \delta_{i,\ell}\hat{Q}^V_{k,j}\,.
\end{aligned} \tag{7}$$