Consider a QFT of a charged Dirac field $\Psi_c(x)$, a neutral Dirac field $\Psi_n(x)$, a charged scalar field $\Phi(x)$ and the electromagnetic field $A^\mu(x)$. Classically — and at the tree level of the quantum theory,

$$\begin{aligned}\mathcal{L} \;=\; & -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; D_\mu\Phi^* D^\mu\Phi \;-\; m_s^2\Phi^*\Phi \;+\; \overline{\Psi}_c(i\not{D}-m_c)\Psi_c \;+\; \overline{\Psi}_n(i\not{\partial}-m_n)\Psi_n \\ & +\; g\Phi\overline{\Psi}_c\Psi_n \;+\; g\Phi^*\overline{\Psi}_n\Psi_c \;-\; \tfrac{1}{4}\lambda\bigl(\Phi^*\Phi\bigr)^2.\end{aligned} \tag{1}$$

1. First, a few simple questions:

   (a) Are there any *renormalizable* couplings one may add to the Lagrangian (1) without breaking any symmetries of the theory? Explain your answer.

   (b) Write down the renormalized Lagrangian of the quantum field theory — including all the counterterms needed to cancel the divergences — and spell out the Feynman rules of the renormalized perturbation theory. Note that some counterterms are related to each other by symmetries and/or Ward–Takahashi identities; make such relations explicit.

   (c) Describe the renormalization conditions that determine the finite parts of all the counterterms.

2. Now comes the hard part: Calculate the infinite parts of all the independent counterterms at the one-loop order of the perturbation theory. To save time, rely on the WT identities: if two counterterms are related by such an identity, calculate the simpler one.

   For each independent counterterm, start by drawing all the relevant Feynman diagrams. If a diagram was evaluated in any of the homeworks, don't waste your time copying the posted solution, just quote the result and move on to the next diagram. Ditto for the diagrams evaluated in class. For the remaining diagrams, use dimensional regularization and work hard.

Note: The net physical amplitudes are gauge invariant, but the individual loop diagrams and the counterterms depend on the gauge you work in. To be consistent, one must use the same gauge in all calculations.

For the purpose of this problem, you may use Feynman gauge or any other gauge you like, but you must use the same gauge for all photon propagators in all the diagrams throughout the whole problem. Even the diagrams you use to calculate different counterterms must be evaluated in the same gauge.

- ⋆ For extra credit, calculate the finite parts of the counterterms. To simplify various integrals over the Feynman parameters, assume $m_n \approx m_c \gg m_s$.

3. Finally, consider the electromagnetic form factors $F_1(q^2)$ and $F_2(q^2)$ of the *neutral* Dirac field $\Psi_n$.

   (a) Draw *all* Feynman diagrams contributing to these form factors at the one-loop level. Do any of the counterterms of the renormalized perturbation theory contribute at this level? Explain your answer.

   (b) Evaluate the diagrams and write them down as as integrals over the Feynman parameters. Allow for generic masses $m_n$, $m_c$ and $m_s$ and any $q^2$.

   (c) Verify that $F_1(q^2 = 0) = 0$ and hence the radiative corrections do not 'endow' the $\Psi_n$ field with a net electric charge. There is however a non-trivial electric charge density, as witnessed by $F_1(q^2) \neq 0$ for $q^2 \neq 0$,

   (d) Finally, assume again $m_n \approx m_c \gg m_s$ and calculate the magnetic dipole moment of the neutral fermion.