

1. According to the Noether theorem, a translationally invariant system of classical fields ϕ_a has a conserved stress-energy tensor

$$T_{\text{Noether}}^{\mu\nu} = \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\nu \phi^a - g^{\mu\nu} \mathcal{L}. \quad (1)$$

Actually, to assure the symmetry of the stress-energy tensor, $T^{\mu\nu} = T^{\nu\mu}$ (which is necessary for the angular momentum conservation), one sometimes has to add a total divergence,

$$T^{\mu\nu} = T_{\text{Noether}}^{\mu\nu} + \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu}, \quad (2)$$

where $\mathcal{K}^{[\lambda\mu]\nu}$ is some 3-index Lorentz tensor antisymmetric in its first two indices.

- (a) Show that regardless of the specific form of $\mathcal{K}^{[\lambda\mu]\nu}(\phi, \partial\phi)$,

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= \partial_\mu T_{\text{Noether}}^{\mu\nu} = (\text{hopefully}) = 0 \\ P_{\text{net}}^\mu &\equiv \int d^3\mathbf{x} T^{0\mu} = \int d^3\mathbf{x} T_{\text{Noether}}^{0\mu}. \end{aligned} \quad (3)$$

For the scalar fields, real or complex, $T_{\text{Noether}}^{\mu\nu}$ is properly symmetric and one simply has $T^{\mu\nu} = T_{\text{Noether}}^{\mu\nu}$. Unfortunately, the situation is more complicated for the vector, tensor or spinor fields. To illustrate the problem, consider the free electromagnetic fields described by the Lagrangian

$$\mathcal{L}(A_\mu, \partial_\nu A_\mu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (4)$$

where A_μ is a real vector field and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

- (b) Write down $T_{\text{Noether}}^{\mu\nu}$ for the free electromagnetic fields and show that it is neither symmetric nor gauge invariant.
- (c) The properly symmetric — and also gauge invariant — stress-energy tensor for the free electromagnetism is

$$T_{\text{EM}}^{\mu\nu} = -F^{\mu\lambda} F^\nu{}_\lambda + \frac{1}{4} g^{\mu\nu} F_{\kappa\lambda} F^{\kappa\lambda}. \quad (5)$$

Show that this expression indeed has form (2) for some $\mathcal{K}^{[\lambda\mu]\nu}$.

(d) Write down the components of the stress-energy tensor (5) in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density and pressure.

2. Now consider the electromagnetic fields coupled to the electric current J^μ of some charged “matter” fields. Because of this coupling, only the *net* energy-momentum of the whole field system should be conserved, but not the separate P_{EM}^μ and P_{mat}^μ . Consequently, we should have

$$\partial_\mu T_{\text{net}}^{\mu\nu} = 0 \quad \text{for} \quad T_{\text{net}}^{\mu\nu} = T_{\text{EM}}^{\mu\nu} + T_{\text{mat}}^{\mu\nu} \quad (6)$$

but generally $\partial_\mu T_{\text{EM}}^{\mu\nu} \neq 0$ and $\partial_\mu T_{\text{mat}}^{\mu\nu} \neq 0$.

(a) Use Maxwell’s equations to show that

$$\partial_\mu T_{\text{EM}}^{\mu\nu} = -F^{\nu\lambda} J_\lambda \quad (7)$$

and therefore any system of charged matter fields should have its stress-energy tensor related to the electric current J_λ according to

$$\partial_\mu T_{\text{mat}}^{\mu\nu} = +F^{\nu\lambda} J_\lambda. \quad (8)$$

Now consider a specific example a complex scalar field of charge $q \neq 0$ coupled to the EM fields:

$$\mathcal{L}_{\text{net}} = D^\mu \Phi^* D_\mu \Phi - m^2 \Phi^* \Phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \quad (9)$$

where

$$D_\mu \Phi = (\partial_\mu + iqA_\mu)\Phi \quad \text{and} \quad D_\mu \Phi^* = (\partial_\mu - iqA_\mu)\Phi^* \quad (10)$$

are the *covariant* derivatives of the scalar fields.

(b) Verify the gauge invariance of the electric current

$$J^\mu = -\frac{\partial \mathcal{L}}{\partial A_\mu} \quad (11)$$

by expressing it in terms of the scalar fields and their covariant derivatives (10).

- (c) Write down equations of motion for the scalar fields Φ and Φ^* and use them to verify the current conservation $\partial_\mu J^\mu = 0$.
- (d) Write down the Noether stress-energy tensor for the whole field system and show that

$$T_{\text{net}}^{\mu\nu} \equiv T_{\text{EM}}^{\mu\nu} + T_{\text{mat}}^{\mu\nu} = T_{\text{Noether}}^{\mu\nu} + \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu} \quad (12)$$

where $\mathcal{K}^{[\lambda\mu]\nu}$ is the same function of EM fields as in the free EM case (1.c), $T_{\text{EM}}^{\mu\nu}$ is exactly as in eq. (5), and

$$T_{\text{mat}}^{\mu\nu} = D^\mu \Phi^* D^\nu \Phi + D^\nu \Phi^* D^\mu \Phi - g^{\mu\nu} (D_\lambda \Phi^* D^\lambda \Phi - m^2 \Phi^* \Phi). \quad (13)$$

Hint: In the presence of an electric current J^μ , the $\partial_\lambda \mathcal{K}^{[\lambda\mu]\nu}$ correction to the electromagnetic stress-energy tensor contains an extra $J^\mu A^\nu$ term. This term is important for obtaining a gauge-invariant stress-energy tensor (13) for the scalar field.

- (e) Use the scalar fields' equations of motion and non-commutativity of the covariant derivatives

$$[D_\mu, D_\nu] \Phi = iqF_{|\mu\nu} \Phi, \quad [D_\mu, D_\nu] \Phi^* = -iqF_{|\mu\nu} \Phi^* \quad (14)$$

to verify eq. (8).