

1. An operator acting on identical bosons can be described in terms of  $N$ -particle wave functions (the *first-quantized* formalism) or in terms of creation and annihilation operators in the Fock space (the *second-quantized* formalism). This exercise is about converting the operators from one formalism to another.

In class, we defined the creation and annihilation operators  $\hat{a}_\alpha^\dagger$  and  $\hat{a}_\alpha$  in the occupation number basis according to

$$\hat{a}_\alpha^\dagger |\{n_\beta\}\rangle = \sqrt{n_\alpha + 1} |\{n'_\beta = n_\beta + \delta_{\alpha\beta}\}\rangle, \quad (1)$$

$$\hat{a}_\alpha |\{n_\beta\}\rangle = \sqrt{n_\alpha} |\{n'_\beta = n_\beta - \delta_{\alpha\beta}\}\rangle \quad (\text{or } 0 \text{ when } n_\alpha = 0). \quad (2)$$

We also wrote the wave functions of the  $|\{n_\beta\}\rangle$  states: Let  $N = \sum_\beta n_\beta$  is the number of particles and let  $|\alpha_1, \dots, \alpha_N\rangle = |\{n_\beta\}\rangle$ ; then

$$\varphi_{\alpha_1, \dots, \alpha_N}(\mathbf{x}_1, \dots, \mathbf{x}_N) = \frac{1}{\sqrt{C_{\alpha_1, \dots, \alpha_N}}} \sum_{\substack{\text{distinct permutations} \\ (\tilde{\alpha}_1, \dots, \tilde{\alpha}_N) \text{ of } (\alpha_1, \dots, \alpha_N)}} \varphi_{\tilde{\alpha}_1}(\mathbf{x}_1) \cdots \varphi_{\tilde{\alpha}_N}(\mathbf{x}_N), \quad (3)$$

where  $C_{\alpha_1, \dots, \alpha_N}$  is the number of distinct permutations.

Our first task here is to derive the wave-function action of the creation and annihilation operators (1) and (2) using eq. (3).

- (a) Consider an  $N$ -particle state  $|N, \Psi\rangle$  with a completely generic totally-symmetric wave function  $\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N)$ . Show that the  $(N - 1)$ -particle state  $|(N - 1), \Psi'\rangle = \hat{a}_\gamma |N, \Psi\rangle$  has wave function

$$\Psi'(\mathbf{x}_1, \dots, \mathbf{x}_{N-1}) = \sqrt{N} \int d^3\mathbf{x}_N \varphi_\gamma^*(\mathbf{x}_N) \Psi(\mathbf{x}_1, \dots, \mathbf{x}_{N-1}, \mathbf{x}_N). \quad (4)$$

Hint: First verify this formula for  $\Psi$  of the form (3), and then generalize to arbitrary (but totally-symmetric)  $\Psi$  by linearity.

- (b) Next, show that the  $(N + 1)$ -particle state  $|(N + 1), \Psi''\rangle = \hat{a}_\gamma^\dagger |N, \Psi\rangle$  has wave function

$$\Psi''(\mathbf{x}_1, \dots, \mathbf{x}_{N+1}) = \frac{1}{\sqrt{N + 1}} \sum_{i=1}^{N+1} \varphi_\gamma(\mathbf{x}_i) \Psi(\mathbf{x}_1, \dots, \cancel{\mathbf{x}_i}, \dots, \mathbf{x}_{N+1}). \quad (5)$$

Hint: Use the fact that  $\hat{a}_\gamma^\dagger$  is the hermitian conjugate of  $\hat{a}_\gamma$ .

Now consider a one-body operator  $\hat{A}_1$ . In the first-quantized formalism  $\hat{A}_{\text{tot}}$  acts on  $N$ -particle states according to

$$\hat{A}_{\text{tot}}^{(1)} = \sum_{i=1}^N \hat{A}_1(i^{\text{th}} \text{ particle}) \quad (6)$$

while in the second-quantized formalism it becomes

$$\hat{A}_{\text{tot}}^{(2)} = \sum_{\alpha, \beta} \langle \alpha | \hat{A}_1 | \beta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta. \quad (7)$$

(c) Use eq. (4) and/or eq. (5) to verify that for any two  $N$ -particle states  $\langle N, \Psi_1 |$  and  $|N, \Psi_2\rangle$

$$\langle N, \Psi_1 | \hat{A}_{\text{tot}}^{(1)} |N, \Psi_2\rangle = \langle N, \Psi_1 | \hat{A}_{\text{tot}}^{(2)} |N, \Psi_2\rangle. \quad (8)$$

Hint: Use  $\hat{A}_1 = \sum_{\alpha, \beta} |\alpha\rangle \langle \alpha | \hat{A}_1 | \beta \rangle \langle \beta |$ .

Next, consider a two-body operator  $\hat{B}_2$  which acts in the first-quantized formalism according to

$$\hat{B}_{\text{tot}}^{(1)} = \frac{1}{2} \sum_{i \neq j} \hat{B}_2(i^{\text{th}} \text{ and } j^{\text{th}} \text{ particles}) \quad (9)$$

and in the second-quantized formalism according to

$$\hat{B}_{\text{tot}}^{(2)} = \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} (\langle \alpha | \otimes \langle \beta |) \hat{B}_2(|\gamma\rangle \otimes |\delta\rangle) \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta. \quad (10)$$

(d) Again, show that for any two  $N$ -particle states  $\langle N, \Psi_1 |$  and  $|N, \Psi_2\rangle$

$$\langle N, \Psi_1 | \hat{B}_{\text{tot}}^{(1)} |N, \Psi_2\rangle = \langle N, \Psi_1 | \hat{B}_{\text{tot}}^{(2)} |N, \Psi_2\rangle. \quad (11)$$

2. Next, an exercise in bosonic commutation relations

$$[\hat{a}_\alpha, \hat{a}_\beta] = 0, \quad [\hat{a}_\alpha^\dagger, \hat{a}_\beta^\dagger] = 0, \quad [\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta}. \quad (12)$$

(a) Calculate the commutators  $[\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger]$ ,  $[\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\delta]$  and  $[\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger \hat{a}_\delta]$ .

- (b) Consider three one-body operators  $\hat{A}_1$ ,  $\hat{B}_1$ , and  $\hat{C}_1$ . Let us define the corresponding second-quantized operators  $\hat{A}_{\text{tot}}^{(2)}$ ,  $\hat{B}_{\text{tot}}^{(2)}$ , and  $\hat{C}_{\text{tot}}^{(2)}$  according to eq. (7).

Show that if  $\hat{C}_1 = [\hat{A}_1, \hat{B}_1]$  then  $\hat{C}_{\text{tot}}^{(2)} = [\hat{A}_{\text{tot}}^{(2)}, \hat{B}_{\text{tot}}^{(2)}]$ .

- (c) Next, calculate the commutator  $[\hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta, \hat{a}_\mu^\dagger \hat{a}_\nu]$ .

- (d) Finally, let  $\hat{A}_1$  be a one-body operator, let  $\hat{B}_2$  and  $\hat{C}_2$  be two-body operators, and let  $\hat{A}_{\text{tot}}^{(2)}$ ,  $\hat{B}_{\text{tot}}^{(2)}$ , and  $\hat{C}_{\text{tot}}^{(2)}$  be the corresponding second-quantized operators according to eqs. (7) and (10).

Show that if  $\hat{C}_2 = \left[ \left( \hat{A}_1(1^{\text{st}}) + \hat{A}_1(2^{\text{nd}}) \right), \hat{B}_2 \right]$  then  $\hat{C}_{\text{tot}}^{(2)} = [\hat{A}_{\text{tot}}^{(2)}, \hat{B}_{\text{tot}}^{(2)}]$ .

3. The rest of this homework is about coherent states of harmonic oscillators and free quantum fields. Let us start with a harmonic oscillator  $\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a}$ .

- (a) For any complex number  $\xi$  we define a *coherent state*  $|\xi\rangle \stackrel{\text{def}}{=} \exp(\xi \hat{a}^\dagger - \xi^* \hat{a}) |0\rangle$ . Show that

$$|\xi\rangle = e^{-|\xi|^2/2} e^{\xi \hat{a}^\dagger} |0\rangle \quad \text{and} \quad \hat{a} |\xi\rangle = \xi |\xi\rangle. \quad (13)$$

- (b) Calculate the uncertainties  $\Delta q$  and  $\Delta p$  for a coherent state  $|\xi\rangle$  and verify their minimality:  $\Delta q \Delta p = \frac{1}{2} \hbar$ . Also, verify  $\delta n = \sqrt{\bar{n}}$  where  $\bar{n} \stackrel{\text{def}}{=} \langle \hat{n} \rangle = |\xi|^2$ .

Hint: use  $\hat{a} |\xi\rangle = \xi |\xi\rangle$  and  $\langle \xi | \hat{a}^\dagger = \xi^* \langle \xi |$ .

- (c) Consider time-dependent coherent states  $|\xi(t)\rangle$ . Show that for  $\xi(t) = \xi_0 e^{-i\omega t}$ , the state  $|\xi(t)\rangle$  satisfies the time-dependent Schrödinger equation  $i\hbar \frac{d}{dt} |\xi(t)\rangle = \hat{H} |\xi(t)\rangle$ .

- (d) The coherent states are not quite orthogonal to each other. Calculate their overlap  $\langle \eta | \xi \rangle$ .

Now consider coherent states of multi-oscillator systems and hence quantum fields. In particular, let us focus on the creation and annihilation fields  $\hat{\Psi}^\dagger(\mathbf{x})$  and  $\hat{\Psi}(\mathbf{x})$  for non-relativistic spinless bosons.

- (e) Generalize (a) and construct coherent states  $|\Phi\rangle$  which satisfy

$$\hat{\Psi}(\mathbf{x}) |\Phi\rangle = \Phi(\mathbf{x}) |\Phi\rangle \quad (14)$$

for any given classical complex field  $\Phi(\mathbf{x})$ .

(f) Show that for any such coherent state,  $\Delta N = \sqrt{\bar{N}}$  where

$$\bar{N} \stackrel{\text{def}}{=} \langle \Phi | \hat{N} | \Phi \rangle = \int d\mathbf{x} |\Phi(\mathbf{x})|^2. \quad (15)$$

(g) Let

$$\hat{H} = \int d\mathbf{x} \left( \frac{\hbar^2}{2M} \nabla \hat{\Psi}^\dagger \cdot \nabla \hat{\Psi} + V(\mathbf{x}) \hat{\Psi}^\dagger \hat{\Psi} \right)$$

and show that for any classical field configuration  $\Phi(\mathbf{x}, t)$  that satisfies the classical field equation

$$i\hbar \frac{\partial}{\partial t} \Phi(\mathbf{x}, t) = \left( -\frac{\hbar^2}{2M} \nabla^2 + V(\mathbf{x}) \right) \Phi(\mathbf{x}, t),$$

the time-dependent coherent state  $|\Phi\rangle$  satisfies the true Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\Phi\rangle = \hat{H} |\Phi\rangle. \quad (16)$$

(h) Finally, show that the quantum overlap  $|\langle \Phi_1 | \Phi_2 \rangle|^2$  between two different coherent states is exponentially small for any *macroscopic* difference  $\delta\Phi(\mathbf{x}) = \Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})$  between the two field configurations.