

1. Back in homework#2 we developed Hamiltonian formalism for a massive vector field $A^\mu(x)$. Upon quantization, the 3-vector field $\mathbf{A}(x)$ and its canonical conjugate $-\mathbf{E}(x)$ become quantum fields subject to equal-time commutation relations

$$[\hat{A}^i(\mathbf{x}), \hat{A}^j(\mathbf{y})] = 0, \quad [\hat{E}^i(\mathbf{x}), \hat{E}^j(\mathbf{y})] = 0, \quad [\hat{A}^i(\mathbf{x}), \hat{E}^j(\mathbf{y})] = -i\delta^{ij}\delta^{(3)}(\mathbf{x} - \mathbf{y}) \quad (1)$$

($\hbar = 1, c = 1$ units) governed by the free Hamiltonian

$$\hat{H} = \int d^3\mathbf{x} \left(\frac{1}{2}\hat{\mathbf{E}}^2 + \frac{(\nabla \cdot \hat{\mathbf{E}})^2}{2m^2} + \frac{1}{2}(\nabla \times \hat{\mathbf{A}})^2 + \frac{1}{2}m^2\hat{\mathbf{A}}^2 \right) \quad (2)$$

(we assume $J^\mu = 0$). For the non-dynamical A^0 field, its time-independent equation of motion becomes an operatorial identity

$$\hat{A}^0(x) = -\frac{\nabla \cdot \hat{\mathbf{E}}(x)}{m^2}. \quad (3)$$

The purpose of the present exercise is to expand fields in terms of creation and annihilation operators $\hat{a}_{\mathbf{k},\lambda}^\dagger$ and $\hat{a}_{\mathbf{k},\lambda}$ where λ labels three different polarization states of a vector particle (spin = 1). Generally, bases for polarization states correspond to \mathbf{k} -dependent complex bases $\mathbf{e}_\lambda(\mathbf{k})$ for ordinary 3-vectors,

$$\mathbf{e}_\lambda(\mathbf{k}) \cdot \mathbf{e}_{\lambda'}^*(\mathbf{k}) = \delta_{\lambda,\lambda'} \quad (4)$$

Of particular convenience is the helicity basis of eigenvectors of the vector product $i\mathbf{k} \times$, namely

$$i\mathbf{k} \times \mathbf{e}_\lambda(\mathbf{k}) = \lambda|\mathbf{k}|\mathbf{e}_\lambda(\mathbf{k}), \quad \lambda = -1, 0, +1. \quad (5)$$

By convention, the overall phases of the helicity eigenvectors are chosen such that

$$\mathbf{e}_0(\mathbf{k}) = \frac{\mathbf{k}}{|\mathbf{k}|}, \quad \mathbf{e}_\lambda^*(\mathbf{k}) = (-1)^\lambda \mathbf{e}_{-\lambda}(\mathbf{k}), \quad \mathbf{e}_\lambda(-\mathbf{k}) = -\mathbf{e}_\lambda^*(+\mathbf{k}). \quad (6)$$

Combining Fourier transform with a basis decomposition, we have

$$\hat{\mathbf{A}}(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\lambda} e^{i\mathbf{k}\mathbf{x}} \mathbf{e}_{\lambda}(\mathbf{k}) \hat{A}_{\mathbf{k},\lambda}, \quad \hat{A}_{\mathbf{k},\lambda} = \int d^3\mathbf{x} e^{-i\mathbf{k}\mathbf{x}} \mathbf{e}_{\lambda}^*(\mathbf{k}) \cdot \hat{\mathbf{A}}(\mathbf{x}) \quad (7)$$

and ditto for the $\hat{\mathbf{E}}(\mathbf{x})$ field and its $\hat{E}_{\mathbf{k},\lambda}$ modes.

(a) Show that $\hat{A}_{\mathbf{k},\lambda}^{\dagger} = -\hat{A}_{-\mathbf{k},\lambda}$, $\hat{E}_{\mathbf{k},\lambda}^{\dagger} = -\hat{E}_{-\mathbf{k},\lambda}$, and derive the equal-time commutation relations for the $\hat{A}_{\mathbf{k},\lambda}$ and $\hat{E}_{\mathbf{k},\lambda}$ operators.

(b) Show that

$$\hat{H} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\lambda} \left(\frac{C_{\mathbf{k},\lambda}}{2} \hat{E}_{\mathbf{k},\lambda}^{\dagger} \hat{E}_{\mathbf{k},\lambda} + \frac{\omega_{\mathbf{k}}^2}{2C_{\mathbf{k},\lambda}} \hat{A}_{\mathbf{k},\lambda}^{\dagger} \hat{A}_{\mathbf{k},\lambda} \right) \quad (8)$$

where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$ and $C_{\mathbf{k},\lambda} = 1 + \delta_{\lambda,0}(\mathbf{k}^2/m^2)$.

(c) Define creation and annihilation operators according to

$$\hat{a}_{\mathbf{k},\lambda} = \frac{\omega_{\mathbf{k}} \hat{A}_{\mathbf{k},\lambda} - iC_{\mathbf{k},\lambda} \hat{E}_{\mathbf{k},\lambda}}{\sqrt{C_{\mathbf{k},\lambda}}}, \quad \hat{a}_{\mathbf{k},\lambda}^{\dagger} = \frac{\omega_{\mathbf{k}} \hat{A}_{\mathbf{k},\lambda}^{\dagger} + iC_{\mathbf{k},\lambda} \hat{E}_{\mathbf{k},\lambda}^{\dagger}}{\sqrt{C_{\mathbf{k},\lambda}}} \quad (9)$$

and verify that they satisfy (relativistically-normalized) equal-time bosonic commutation relations.

(d) Show that

$$\hat{H} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \sum_{\lambda} \omega_{\mathbf{k}} \hat{a}_{\mathbf{k},\lambda}^{\dagger} \hat{a}_{\mathbf{k},\lambda} + \text{const.} \quad (10)$$

(e) Next, consider the time dependence of the free vector field and show that

$$\hat{\mathbf{A}}(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \sum_{\lambda} \sqrt{C_{\mathbf{k},\lambda}} \left(e^{-i\mathbf{k}\mathbf{x}} \mathbf{e}_{\lambda}(\mathbf{k}) \hat{a}_{\mathbf{k},\lambda}(0) + e^{+i\mathbf{k}\mathbf{x}} \mathbf{e}_{\lambda}^*(\mathbf{k}) \hat{a}_{\mathbf{k},\lambda}^{\dagger}(0) \right)_{k^0=+\omega_{\mathbf{k}}} \quad (11)$$

(f) Write down a similar formula for the $\hat{A}^0(\mathbf{x}, t)$ (use eq. (3)). Together with the previous

result, you should get

$$\hat{A}_\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_{\lambda} \left(e^{-ikx} f_\mu(\mathbf{k}, \lambda) \hat{a}_{\mathbf{k},\lambda}(0) + e^{+ikx} f_\mu^*(\mathbf{k}, \lambda) \hat{a}_{\mathbf{k},\lambda}^\dagger(0) \right)_{k^0 = +\omega_{\mathbf{k}}} \quad (12)$$

where

$$f^\mu(\mathbf{k}, \lambda) = \begin{cases} (0, \mathbf{e}_\lambda(\mathbf{k})) & \text{for } \lambda = \pm 1, \\ \left(\frac{|\mathbf{k}|}{m}, \frac{\omega_{\mathbf{k}}}{m} \frac{\mathbf{k}}{|\mathbf{k}|} \right) & \text{for } \lambda = 0. \end{cases} \quad (13)$$

Note that the 4-vectors $f^\mu(\mathbf{k}, \lambda)$ are basically the purely-spatial vectors $\mathbf{e}_\lambda(\mathbf{k})$ Lorentz-boosted into the moving particle's frame. In particular, for all (\mathbf{k}, λ) , $f^\mu f_\mu^* = -1$ and $f^\mu k_\mu = 0$.

- (g) Finally, verify that the vector field (12) satisfies the free equations of motion $\partial_\mu \hat{A}^\mu(x) = 0$ and $(\partial^2 + m^2)\hat{A}^\mu(x) = 0$.

2. Now consider the Feynman propagator for the massive vector field.

- (a) First, a lemma: Show that

$$\sum_{\lambda} f^\mu(\mathbf{k}, \lambda) f^{\nu*}(\mathbf{k}, \lambda) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}. \quad (14)$$

- (b) Next, show that

$$\begin{aligned} \langle 0 | \hat{A}^\mu(x) \hat{A}^\nu(y) | 0 \rangle &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}} \left[\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \right) e^{-ik(x-y)} \right]_{k^0 = +\omega_{\mathbf{k}}} \\ &= \left(-g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{m^2} \right) D(x-y). \end{aligned} \quad (15)$$

- (c) Finally, the Feynman propagator: Show that

$$\begin{aligned} G_F^{\mu\nu} &\equiv \langle 0 | \mathbf{T}^* \hat{A}^\mu(x) \hat{A}^\nu(y) | 0 \rangle = \left(-g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{m^2} \right) G_F(x-y) \\ &= \int \frac{d^4\mathbf{k}}{(2\pi)^4} \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \right) \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i0} \end{aligned} \quad (16)$$

where

$$\mathbf{T}^* \hat{A}^\mu(x) \hat{A}^\nu(y) = \mathbf{T} \hat{A}^\mu(x) \hat{A}^\nu(y) + i \delta^{\mu 0} \delta^{\nu 0} \delta^{(4)}(x - y). \quad (17)$$

For the explanation of the \mathbf{T}^* modification of the time-ordered product of vector fields, please see *Quantum Field Theory* by Claude Itzykson and Jean-Bernard Zuber.

3. Finally, an exercise in Dirac's γ matrices. Assume the anti-commutation relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \quad (18)$$

and the definition

$$S^{\mu\nu} = -S^{\nu\mu} \stackrel{\text{def}}{=} \frac{1}{4i} [\gamma^\mu, \gamma^\nu] \quad (19)$$

but do not assume the specific form of the 4×4 matrices γ^μ and $S^{\mu\nu}$.

(a) Show that

$$[S^{\kappa\lambda}, \gamma^\mu] = -ig^{\lambda\mu} \gamma^\kappa + ig^{\kappa\mu} \gamma^\lambda \quad (20)$$

and

$$[S^{\kappa\lambda}, S^{\mu\nu}] = ig^{\lambda\mu} S^{\kappa\nu} - ig^{\lambda\nu} S^{\kappa\mu} - ig^{\kappa\mu} S^{\lambda\nu} + ig^{\kappa\nu} S^{\lambda\mu}. \quad (21)$$

(b) Calculate $\{\gamma^\rho, \gamma^\lambda \gamma^\mu \gamma^\nu\}$, $[\gamma^\rho, \gamma^\kappa \gamma^\lambda \gamma^\mu \gamma^\nu]$ and $[S^{\rho\sigma}, \gamma^\lambda \gamma^\mu \gamma^\nu]$.

(c) Show that $\gamma^\alpha \gamma_\alpha = 4$, $\gamma^\alpha \gamma^\nu \gamma_\alpha = -2\gamma^\nu$, $\gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha = 4g^{\mu\nu}$ and $\gamma^\alpha \gamma^\lambda \gamma^\mu \gamma^\nu \gamma_\alpha = -2\gamma^\nu \gamma^\mu \gamma^\lambda$.

Hint: use $\gamma^\alpha \gamma^\nu = 2g^{\nu\alpha} - \gamma^\nu \gamma^\alpha$ repeatedly.

Continuous Lorentz transforms obtain from integrating infinite sequences of infinitesimal transforms $X'^\mu = X^\mu + \epsilon \Theta^\mu_\nu X^\nu$ where $\Theta_{\mu\nu} = -\Theta_{\nu\mu}$. Altogether, a finite continuous transform acts as $X'^\mu = L^\mu_\nu X^\nu$ where

$$L = \exp(\Theta), \quad i. e., \quad L^\mu_\nu = \delta^\mu_\nu + \Theta^\mu_\nu + \frac{1}{2} \Theta^\mu_\lambda \Theta^\lambda_\nu + \frac{1}{6} \Theta^\mu_\kappa \Theta^\kappa_\lambda \Theta^\lambda_\nu + \dots \quad (22)$$

(d) Let L be a Lorentz transform of the form (22), and let $M(L) = \exp(-\frac{i}{2} \theta_{\alpha\beta} S^{\alpha\beta})$.

Show that $M^{-1}(L) \gamma^\mu M(L) = L^\mu_\nu \gamma^\nu$.