1. Consider the matrix $\gamma^5 \stackrel{\rm def}{=} i\gamma^0\gamma^1\gamma^2\gamma^3$.

   (a) Show that γ^5 anticommutes with each of the γ^μ matrices, $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$.

   (b) Show that γ^5 is hermitian and that $(\gamma^5)^2 = 1$.

   (c) Show that $\gamma^5 = (-i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ and $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} = -i\epsilon^{\kappa\lambda\mu\nu}\,\gamma^5$.

   (d) Show that $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = i\epsilon^{\kappa\lambda\mu\nu}\,\gamma_\kappa\gamma^5$.

   (e) Show that any 4×4 matrix Γ is a unique linear combination of the following 16 matrices: 1, γ^μ, $\gamma^{[\mu}\gamma^{\nu]}$, $\gamma^5\gamma^\mu$ and γ^5.

   Conventions: $\epsilon^{0123} = +1$, $\epsilon_{0123} = -1$, $\gamma^{[\mu}\gamma^{\nu]} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$,
   $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = \frac{1}{6}(\gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\lambda\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\mu\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\lambda)$,
   and ditto for the $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]}$.

2. Consider bilinear products of a Dirac field $\Psi(x)$ and its conjugate $\overline{\Psi}(x)$. Generally, such products have form $\overline{\Psi}\Gamma\Psi$ where Γ is one of 16 matrices discussed in **1**.(e); altogether, we have

$$S = \overline{\Psi}\Psi, \quad V^\mu = \overline{\Psi}\gamma^\mu\Psi, \quad T^{\mu\nu} = \overline{\Psi}i\gamma^{[\mu}\gamma^{\nu]}\Psi, \quad A^\mu = \overline{\Psi}\gamma^5\gamma^\mu\Psi \quad \text{and} \quad P = \overline{\Psi}i\gamma^5\Psi. \tag{1}$$

   (a) Show that all the bilinears (1) are Hermitian.
   Hint: First, show that $\left(\overline{\Psi}\Gamma\Psi\right)^\dagger = \overline{\Psi}\overline{\Gamma}\Psi$

   (b) Show that under *continuous* Lorentz symmetries, the S and the P transform as scalars, the V^μ and the A^μ as vectors and the $T^{\mu\nu}$ as an antisymmetric tensor.

   (c) Find the transformation rules of the bilinears (1) under parity (*cf.* problem 2 of the previous set) and show that while S is a true scalar and V is a true (polar) vector, P is a pseudoscalar and A is an axial vector.

   Next, consider the charge-conjugation properties of Dirac bilinears. To avoid operator ordering problems, take $\Psi(x)$ and $\Psi^\dagger(x)$ to be "classical" fermionic fields which *anticommute* with each other, $\Psi_\alpha\Psi^\dagger_\beta = -\Psi^\dagger_\beta\Psi_\alpha$.

(d) In the Weyl convention, $\hat{\mathcal{C}}\hat{\Psi}(x)\hat{\mathcal{C}} = \pm\gamma^2\hat{\Psi}^*(x)$. Show that $\hat{\mathcal{C}}\hat{\bar{\Psi}}\Gamma\hat{\Psi}\hat{\mathcal{C}} = \hat{\bar{\Psi}}\Gamma^c\hat{\Psi}$ where $\Gamma^c = \gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2$.

(e) Calculate Γ^c for all 16 independent matrices Γ and find out which Dirac bilinears are $\mathcal{C}$–even and which are $\mathcal{C}$–odd.

3. Next, an exercise in fermionic creation and annihilation operators and their anticommutation relations,

$$\{\hat{a}_\alpha, \hat{a}_\beta\} = \{\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta\} = 0, \quad \{\hat{a}_\alpha, \hat{a}^\dagger_\beta\} = \delta_{\alpha,\beta}\,. \tag{2}$$

(a) Calculate the commutators $[\hat{a}^\dagger_\alpha\hat{a}_\beta, \hat{a}^\dagger_\gamma]$, $[\hat{a}^\dagger_\alpha\hat{a}_\beta, \hat{a}_\delta]$ and $[\hat{a}^\dagger_\alpha\hat{a}_\beta, \hat{a}^\dagger_\gamma\hat{a}_\delta]$.

(b) Consider two one-body operators $\hat{A}_1$ and $\hat{B}_1$ and let $\hat{C}_1$ be their commutator, $\hat{C}_1 = [\hat{A}_1, \hat{B}_1]$. Let $\hat{A}$ be the second-quantized forms of $\hat{A}_{\text{tot}}$,

$$\hat{A} = \sum_{\alpha,\beta} \langle\alpha|\,\hat{A}_1\,|\beta\rangle\;\hat{a}^\dagger_\alpha\hat{a}_\beta\,, \tag{3}$$

and ditto for the second-quantized $\hat{B}$ and $\hat{C}$.

Verify that $[\hat{A}, \hat{B}] = \hat{C}$.

(c) Calculate the commutator $[\hat{a}^\dagger_\mu\hat{a}_\nu, \hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\delta]$.

(d) The second quantized form of a two-body additive operator

$$\hat{B}_{\text{tot}} = \tfrac{1}{2}\sum_{i\neq j} \hat{B}_2(i^{\text{th}} \text{ and } j^{\text{th}} \text{ particles})$$

acting on identical fermions is

$$\hat{B} = \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta} \langle\alpha\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\delta\rangle\;\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\delta\hat{a}_\gamma\,. \tag{4}$$

This expression is similar to its bosonic counterpart, but note the reversed order of the annihilation operators $\hat{a}_\delta$ and $\hat{a}_\gamma$.

Consider a one-body operator $\hat{A}_1$ and two two-body operators $\hat{B}_2$ and $\hat{C}_2$. Show that if $\hat{C}_2 = \left[\left(\hat{A}_1(1^{\text{st}}) + \hat{A}_1(2^{\text{nd}})\right), \hat{B}_2\right]$, then the respective second-quantized operators in the fermionic Fock space satisfy $\hat{C} = [\hat{A}, \hat{B}]$.

4. Finally, consider the quantum Dirac fields

$$\begin{aligned}
\hat{\Psi}(x) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \sum_s \left(e^{-ipx} u(\mathbf{p},s) \hat{a}_{\mathbf{p},s} + e^{+ipx} v(\mathbf{p},s) \hat{b}^\dagger_{\mathbf{p},s} \right)_{p^0=+E_\mathbf{p}} , \\
\hat{\overline{\Psi}}(x) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \sum_s \left(e^{-ipx} \bar{v}(\mathbf{p},s) \hat{b}_{\mathbf{p},s} + e^{+ipx} \bar{u}(\mathbf{p},s) \hat{a}^\dagger_{\mathbf{p},s} \right)_{p^0=+E_\mathbf{p}} ,
\end{aligned} \tag{5}$$

where $\hat{a}$, $\hat{b}$, $\hat{a}^\dagger$, and $\hat{b}^\dagger$ are relativistically normalized fermionic annihilation and creation operators, thus

$$\{\hat{a}_{\mathbf{p},s}, \hat{a}^\dagger_{\mathbf{p}',s'}\} = \{\hat{b}_{\mathbf{p},s}, \hat{b}^\dagger_{\mathbf{p}',s'}\} = \delta_{s,s'} \times 2E_\mathbf{p}(2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{p}') \tag{6}$$

while all other anticommutators vanish,

$$\{\hat{a} \,\text{or}\, \hat{b}\,,\, \hat{a} \,\text{or}\, \hat{b}\} = 0, \quad \{\hat{a}^\dagger \,\text{or}\, \hat{b}^\dagger\,,\, \hat{a}^\dagger \,\text{or}\, \hat{b}^\dagger\} = 0, \quad \{\hat{a}, \hat{b}^\dagger\} = \{\hat{b}, \hat{a}^\dagger\} = 0. \tag{7}$$

As discussed in class, the free Dirac Hamiltonian is

$$\hat{H} = \int d^3\mathbf{x} \hat{\overline{\Psi}}(-i\vec{\gamma}\cdot\nabla + m)\hat{\Psi} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \sum_s \left(E_\mathbf{p}\, \hat{a}^\dagger_{\mathbf{p},s}\hat{a}_{\mathbf{p},s} + E_\mathbf{p}\, \hat{b}^\dagger_{\mathbf{p},s}\hat{b}_{\mathbf{p},s} \right) + \text{const.} \tag{8}$$

(a) Derive Dirac field's stress-energy tensor (use Noether theorem) and show that the net mechanical momentum is

$$\hat{\mathbf{P}}_\text{mech} = \int d^3\mathbf{x}\, \hat{\Psi}^\dagger(-i\nabla)\hat{\Psi} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \sum_s \left(\mathbf{p}\, \hat{a}^\dagger_{\mathbf{p},s}\hat{a}_{\mathbf{p},s} + \mathbf{p}\, \hat{b}^\dagger_{\mathbf{p},s}\hat{b}_{\mathbf{p},s} \right). \tag{9}$$

(b) Show that the electric 4-current of the electron field is $J^\mu(x) = -e\overline{\Psi}(x)\gamma^\mu\Psi(x)$ and that the net electric charge operator is

$$\begin{aligned}
\hat{Q} &= -e \int d^3\mathbf{x}\, \hat{\Psi}^\dagger(x)\hat{\Psi}(x) + \text{constant} \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \sum_s \left(-e\, \hat{a}^\dagger_{\mathbf{p},s}\hat{a}_{\mathbf{p},s} + e\, \hat{b}^\dagger_{\mathbf{p},s}\hat{b}_{\mathbf{p},s} \right).
\end{aligned} \tag{10}$$

Note: The constant term in the first line arises from the operator ordering ambiguity when the classical electron field is quantized. It's actual value — which happens to be infinite — is determined by demanding that the vacuum state has zero electric charge.

(c) Finally, consider the net spin of electrons and positrons,

$$\hat{\mathbf{S}}_{\mathrm{net}} \;=\; \int d^3\mathbf{x}\, \hat{\Psi}^\dagger \mathbf{S} \hat{\Psi}. \tag{11}$$

Expand this operator into momentum modes

$$\hat{\mathbf{S}}_{\mathrm{net}} \;=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\,\hat{\mathbf{S}}_{\mathbf{p}} \tag{12}$$

and show that for the non-relativistic modes ($|\mathbf{p}| \ll m$)

$$\hat{\mathbf{S}}_{\mathbf{p}} \;=\; \sum_{s,s'} \xi_s^\dagger \frac{\boldsymbol{\sigma}}{2} \xi_{s'} \times \left(\hat{a}^\dagger_{\mathbf{p},s}\hat{a}_{\mathbf{p},s'} \;+\; \hat{b}^\dagger_{\mathbf{p},s}\hat{b}_{\mathbf{p},s'}\right) \;+\; O(|\mathbf{p}|/m). \tag{13}$$

The relativistic modes with $|\mathbf{p}| \gtrsim O(m)$ are more complicated because of mixing between the spin and the orbital angular momentum.

Hint: Approximate $u(\mathbf{p}, s) \approx u(\mathbf{0}, s)$ and $v(-\mathbf{p}, s) \approx v(\mathbf{0}, s)$ for small $|\mathbf{p}| \ll m$, and use $\eta_s = \sigma_2 \xi_s^*$.

In particle terms, eqs. (8)–(13) mean that the fermionic operator $\hat{a}^\dagger_{\mathbf{p},s}$ creates and $\hat{a}_{\mathbf{p},s}$ annihilates an electron with momentum $\mathbf{p}$, energy $E_{\mathbf{p}} = +\sqrt{m^2 + \mathbf{p}^2}$, spin $= \frac{1}{2}$ and spin state ξ_s, and electric charge $= -e$, while operator $\hat{b}^\dagger_{\mathbf{p},s}$ creates and $\hat{b}_{\mathbf{p},s}$ annihilates a positron with exactly the same momentum, energy, spin and spin state, but electric charge $= +e$.