1. In a Yukawa theory with $M_s > 2m_f$, the scalar particles are unstable against decay into fermion + antifermion pairs.

   Write down the tree-level matrix element $\mathcal{M}(S \to f+\bar{f})$, sum $|\mathcal{M}|^2$ over final particles' spins, and calculate the total decay rate $S \to f + \bar{f}$.

2. Consider a Yukawa theory of two Dirac fields $\Psi_1(x)$ and $\Psi_2(x)$ coupled to the same real scalar field $\Phi(x)$:

$$\begin{aligned}\mathcal{L} &= \overline{\Psi}_1(i\not\partial - m_1)\Psi_1 \;+\; \overline{\Psi}_2(i\not\partial - m_2)\Psi_2 \;+\; \tfrac{1}{2}(\partial_\mu\Phi)^2 \;-\; \tfrac{1}{2}M_s^2\Phi^2 \\ &-\; g_1\Phi\overline{\Psi}_1\Psi_1 \;-\; g_2\Phi\overline{\Psi}_2\Psi_2\,.\end{aligned} \tag{1}$$

   At the tree level, calculate the matrix element, the partial cross-section and the total cross-section for elastic scattering of one type of a fermion off the other type, $f_1 + f_2 \to f_1 + f_2$. Take the initial fermions to be unpolarized and sum over the final fermion's polarizations.

3. Finally, consider the muon decay, $\mu^- \to e^-\bar{\nu}_e\nu_\mu$. Since neutrinos are hard to detect experimentally, the readily measurable quantities for this process are the total muon decay rate $\Gamma_\mu = 1/\tau_\mu$ and the energy distribution of electrons produced by decaying muons; In this exercise, we calculate these quantities from the Fermi theory of weak interactions.

   According to the Fermi theory, the matrix element for muon decay is

$$\left\langle e^-,\bar{\nu}_e,\nu_\mu\right|\mathcal{M}\left|\mu^-\right\rangle = \frac{G_F}{\sqrt{2}}\left[\bar{u}(\nu_\mu)(1-\gamma^5)\gamma^\alpha u(\mu^-)\right]\times\left[\bar{u}(e^-)(1-\gamma^5)\gamma_\alpha v(\bar{\nu}_e)\right]. \tag{2}$$

   The modern Standard Model of particle interactions produces essentially the same answer at the tree level of the perturbation theory.

(a) Sum the absolute square of the amplitude (2) over the final particle spins and average over the initial muon's spin. Show that altogether,

$$\frac{1}{2}\sum_{\substack{\text{all}\\ \text{spins}}} \left|\left\langle e^-, \bar{\nu}_e, \nu_\mu \right| \mathcal{M} \left|\mu^-\right\rangle\right|^2 \;=\; 64 G_F^2 (p_\mu \cdot p_{\bar{\nu}})\,(p_e \cdot p_\nu). \tag{3}$$

The rest of this exercise is the phase space calculation. The following lemma is very useful for three-body decays like $\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$:

(b) Consider a generic three-body decay of some particles of mass M_0 into three particles of respective masses m_1, m_2, and m_3. Show that in the rest frame of the original particle the partial decay rate is given by

$$d\Gamma \;=\; \frac{1}{2M_0} \times \overline{|\mathcal{M}|^2} \times \frac{d^3\Omega}{256\pi^5} \times dE_1\, dE_2\, dE_3\, \delta(E_1 + E_2 + E_3 - M_0) \tag{4}$$

where $d^3\Omega$ refers to three angular variables parameterizing the directions of the three final-state particles relative to some external frame but not affecting the angles between the three momenta. For example, one may use two angles to describe the orientation of the decay plane (the three momenta are coplanar, $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0$) and one more angle to fix the direction of *e.g.*, $\mathbf{p}_1$ in that plane. Altogether, $\int d^3\Omega = 4\pi \times 2\pi = 8\pi^2$.

Also show that when $m_1 = m_2 = m_3 = 0$, the kinematically allowed range of the final particles' energies is given by

$$0 \;\le\; E_1, E_2, E_3 \;\le \tfrac{1}{2} M_0 \qquad \text{while} \quad E_1 + E_2 + E_3 \;=\; M_0, \tag{5}$$

but for the non-zero masses $m_{1,2,3}$ this range is much more complicated.

The electron and the neutrinos are much lighter then the muon, so in most decay events all three final-state particles are ultra-relativistic. This allows us to approximate $m_e \approx m_\nu \approx m_{\bar{\nu}} \approx 0$, which greatly simplifies the last part of this exercise:

(c) Integrate the muon's partial decay rate over the final particle energies and derive first $d\Gamma/dE_e$ and then the total decay rate.