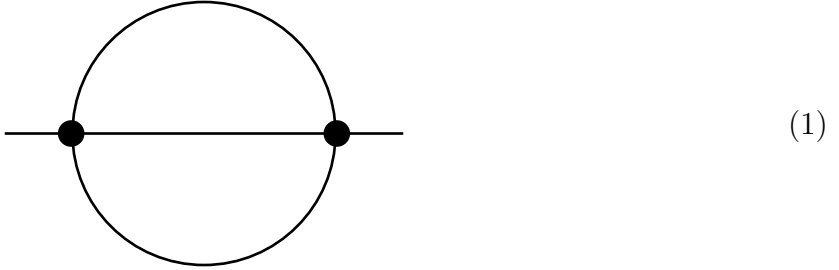


As discussed in class, in the purely scalar field theory, renormalization of the field strength begins at two-loop level. Specifically, the 1PI diagram



provides the leading contribution to the $d\Sigma(p^2)/dp^2$ and hence to the $Z - 1$.

Your task is to evaluate this contribution. This is a difficult calculation, so proceed very carefully.

1. First, use Feynman parameters to write the product of 3 propagators as

$$\prod_{j=1}^3 \frac{i}{q_j^2 - m^2 + i0} = \iiint dx dy dz \delta(x + y + z - 1) \frac{2i^3}{(\mathcal{D})^3} \quad (2)$$

where

$$\mathcal{D} = xq_1^2 + yq_2^2 + zq_3^2 - m^2 + i0. \quad (3)$$

Then substitute $q_3 = p - q_1 = q_2$ and introduce linear combinations k_1 and k_2 of the q_1, q_2 , and p such that $d^4k_1 d^4k_2 = d^4q_1 d^4q_2$ and

$$\mathcal{D} = \alpha k_1^2 + \beta k_2^2 + \gamma p^2 - m^2 + i0 \quad (4)$$

for some (x, y, z) -dependent coefficients α, β, γ .

Warning: Do not set $p^2 = m^2$ at this stage.

2. Express the derivative $d\Sigma(p^2)/dp^2$ in terms of

$$\iint d^4k_1 d^4k_2 \frac{1}{\mathcal{D}^4}. \quad (5)$$

Note that although this integral diverges as $k_{1,2} \rightarrow \infty$, the divergence is logarithmic rather than quadratic.

3. To evaluate the momentum integral (5), first Wick-rotate both momenta k_1 and k_2 to the Euclidean momentum space, and then use dimensional regularization.

Here are some formulæ you will find useful at this stage:

$$\frac{6}{A^4} = \int_0^\infty dt t^3 e^{-At}, \quad (6)$$

$$\int \frac{d^D k}{(2\pi)^D} e^{-ctk^2} = (4\pi ct)^{-D/2}, \quad (7)$$

$$\Gamma(2\epsilon)X^\epsilon = \frac{1}{2\epsilon} - \gamma_E + \frac{1}{2} \log X. \quad (8)$$

4. At this stage, you should arrive at a formula which looks like

$$\begin{aligned} \frac{d\Sigma}{dp^2} = & \iiint dx dy dz \delta(x+y+z-1) F(x,y,z) \times \\ & \times \left\{ \frac{1}{\epsilon} + \log \frac{\mu^2}{m^2} + \text{const} + \log G(x,y,z; p^2/m^2) \right\} \end{aligned} \quad (9)$$

for some rational functions F, G of the Feynman parameters (and in case of G , also of p^2/m^2). Our goal is the field strength renormalization factor

$$Z = \left[1 - \frac{d\Sigma}{dp^2} \right]^{-1} \quad (10)$$

where the derivative is evaluated at $p^2 = M_{\text{ph}}^2$ — the physical mass² of the scalar particle. To the leading approximation, we may let $M_{\text{ph}}^2 \approx m^2$, — and that should simplify the $G(x,y,z)$ function in eq. (9) just a bit.

Nevertheless, the Feynman parameter integral (9) remains a frightful mess. Do not try to evaluate it by hand — it would take way too much time. Instead, you should use Mathematica (or equivalent software). To help it along, replace the (x, y, z) variables with (w, ξ) according to $x = \xi w$, $y = (1 - \xi)w$, $z = 1 - w$, then integrate over the w variable first and over the ξ second. Here is a couple of integrals I did this way you might find useful:

$$\iiint dx dy dz \delta(x + y + z - 1) \times \frac{xyz}{(xy + xz + yz)^3} = \frac{1}{2},$$

$$\iiint dx dy dz \delta(x + y + z - 1) \times \frac{xyz}{(xy + xz + yz)^3} \log \frac{(xy + xz + yz)^3}{(xy + xz + yz - xyz)^2} = -\frac{3}{4}. \quad (11)$$