1. In QED, all physical on-shell amplitudes are gauge-invariant, but the off-shell amplitudes may depend on the gauge-fixing condition used to derive the photon's propagator. Also, the individual loop diagrams are often gauge-dependent, and to compensate for that, the counterterms are also gauge-dependent.

   In class, we have used the Feynman gauge for all calculations. In this problem, we use more general Lorentz-invariant gauges

$$= \frac{-i}{k^2+i0}\left(g^{\mu\nu} + (\xi-1)\frac{k^\mu k^\nu}{k^2+i0}\right) \tag{1}$$

   and study ξ–dependence of various quantities.

   (a) Consider the one-loop vertex correction $ie\Gamma^\mu_{1\,\mathrm{loop}}(p',p)$. Show that for general ξ,

$$\Gamma^\mu_\xi(p',p) = \Gamma^\mu_F(p',p) + e^2(\xi-1)\gamma^\mu \times \int\limits_{\mathrm{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{-i}{(k^2+i0)^2} \tag{2}$$

   where $\Gamma^\mu_F(p',p)$ obtains in the Feynman gauge $\xi = 1$. Note that the second term on the right hand side needs both UV and IR regulators, but you don't need the specifics of such regulators at this point. Hint: Use

$$\frac{1}{\not k + \not p - m} \times \not k = 1 - \frac{1}{\not k + \not p - m} \times (\not p - m) \tag{3}$$

   and ditto for the $\not p'$.

   (b) Use eq. (2) to show that the physical form factors $F_1(q^2)$ and $F_2(q^2)$ are gauge-invariant (at least at the one-loop level) but the δ_1 counterterm is gauge dependent: At the one-loop level

$$\delta_1(\xi) = \delta_1(\mathrm{Feynman}) - e^2(\xi-1) \times \int\limits_{\mathrm{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{-i}{(k^2+i0)^2}\,. \tag{4}$$

(c) Now consider the one-loop correction to the electron's propagator and show that

$$\Sigma_\xi(\not{p}) \;=\; \Sigma_F(\not{p}) \;-\; (\not{p}-m)\times e^2(\xi-1)\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{-i}{(k^2+i0)^2} \;+\; O\big((\not{p}-m)^2\big) \tag{5}$$

where $\Sigma_F(\not{p})$ obtains in the Feynman gauge $\xi = 1$.

(d) Finally, use eq. (5) to show that at the one-loop level

$$\delta_2(\xi) \;=\; \delta_2(\text{Feynman}) \;-\; e^2(\xi-1)\times\int\limits_{\text{reg}}\frac{d^4k}{(2\pi)^4}\,\frac{-i}{(k^2+i0)^2}\,. \tag{6}$$

and hence $\delta_1 = \delta_2$ for any gauge parameter ξ.

2. In the minimal Standard Model, the anomalous magnetic moment $a_\mu = \frac{1}{2}(g-2)$ of the muon has been calculated to an extremely high accuracy of $\Delta_{\text{th}}a_\mu \sim 10^{-11}$ and experimentally measured to an almost as high accuracy of $\Delta_{\text{exp}}a_\mu \approx 80\cdot 10^{-11}$. At present, the theory and the experiment agree with each other, but future refinement may lead to a discrepancy indicating some new physics.

(a) Suppose a non-minimal version of the Standard Model contains a heavy neutral scalar of mass $M_S \simeq 200$ GeV and a Yukawa coupling to the muon spinor, $g\Phi\overline{\Psi}\Psi$.

Calculate the contribution of this field to the muon's magnetic moment at the one-loop level of the perturbation theory. Then use your result to derive an upper limit on the Yukawa coupling g.

(b) A different non-minimal Standard model contains an *axion*, a pseudoscalar field which couples to leptons according to

$$\frac{\partial_\mu\phi}{f_a}\,\overline{\Psi}\gamma^5\gamma^\mu\Psi \;\approx\; \frac{2im_{\text{lepton}}}{f_a}\,\phi\overline{\Psi}\gamma^5\Psi \;+\; \text{a total derivative.} \tag{7}$$

The axion is a pseudo-Goldstone boson resulting from spontaneous breakdown of an axial symmetry at a very high energy scale $f_a \gg 100$ GeV; the symmetry is inexact but very good, and hence the axion is not exactly massless but very light, $M_A \lesssim 1$ MeV.

Calculate the axion's contribution to the muon's magnetic moment at the one-loop level. Then use your result to derive a lower limit on the axion scale f_a.