1. First, a bit of group theory. Consider a generic simple non-abelian Lie group $G$ and its generators $T^a$. The (quadratic) Casimir operator $C_2 = \sum_a T^a T^a$ commutes with all the generators and hence for any irreducible representation $(r)$ of the group, $C_2$ restricted to $(r)$ is simply a unit matrix times a number $C(r)$. In other words, if $T^a_{(r)}$ is a matrix of the generator $T^a$ in the representation $(r)$, then $\sum_a T^a_{(r)} T^a_{(r)} = C(r) \times 1$. For example, for the isospin group $SU(2)$, the irreps are characterized by the isospin $I$ and $C(I) = I(I + 1)$.

(a) By symmetry, for any complete representation $(r)$ of the group,

$$\text{tr}_{(r)}(T^a T^b) \equiv \text{tr}(T^a_{(r)} T^b_{(r)}) = R(r) \delta^{ab}$$

for some coefficient $R(r)$. Show that for any irreducible representation,

$$\frac{R(r)}{C(r)} = \frac{\text{dim}(r)}{\text{dim}(G)}.$$  \hspace{1cm} (2)

In particular, for the $SU(2)$ group, this formula gives $R(I) = \frac{1}{3}I(I + 1)(2I + 1)$.

(b) Suppose the first three generators of $G$ generate an $SU(2)$ subgroup. Show that if a representation $(r)$ of $G$ decomposes into several $SU(2)$ multiplets of isospins $I_1, I_2, \ldots, I_n$, then

$$R(r) = \sum_{i=1}^{n} \frac{1}{3}I_i(I_i + 1)(2I_i + 1).$$  \hspace{1cm} (3)

(c) Now consider the $SU(N)$ group with an obvious $SU(2)$ subgroup of matrices acting on the first two components of a complex $N$-vector. The fundamental representation $(N)$ of the $SU(N)$ decomposes into one doublet and $(N - 2)$ singlets of the $SU(2)$ subgroup, hence

$$R(N) = \frac{1}{2} \quad \text{and} \quad C(N) = \frac{N^2 - 1}{2N}.$$  \hspace{1cm} (4)

Show that the adjoint representation of the $SU(N)$ decomposes into one $SU(2)$
triplet, \(2(N-2)\) doublets and \((N-2)^2\) singlets and hence

\[ R(\text{adj}) = C(\text{adj}) \equiv C(G) = N. \quad (5) \]

Hint: \((N) \times (\overline{N}) = (\text{adj}) + (1)\).

(d) The symmetric and the anti-symmetric 2-index tensors form irreducible representations of the \(SU(N)\) group. Find out the decomposition of these irreps under an \(SU(2) \subset SU(N)\) and calculate their respective \(R\) factors.

2. And now let’s apply group theory to a physical process of quark-antiquark pair production in Quantum ChromoDynamics (QCD). Specifically, let us focus on the \(u\bar{u} \rightarrow d\bar{d}\) process so there is only one tree-level diagram contributing to this process. Draw this diagram and calculate the amplitude, then sum/average the \(|\mathcal{M}|^2\) over both spins and colors of the final/initial particles and calculate the total cross section. For simplicity, you may neglect the quark masses.

Note that the \(u\bar{u} \rightarrow d\bar{d}\) pair production in QCD is very similar to the \(e^-e^+ \rightarrow \mu^-\mu^+\) pair production in QED, so the only new aspect of this problem is summing over the colors.

3. Next, consider a scalar analogue of QCD or more generally a theory of Yang-Mills fields \(A^a_\mu\) and complex scalars \(\Phi_i\) in some representation \((r)\) of the gauge group \(G\).

(a) Write down the Lagrangian and the Feynman rules of this theory.

Next, consider the annihilation process \(\Phi + \Phi^* \rightarrow 2\) gauge bosons. At the tree level, there are four Feynman diagrams contributing to this process.

(b) Draw the diagrams and write down the tree-level annihilation amplitude.

As discussed in class, amplitudes involving the non-abelian gauge fields satisfy a weak form of the Ward identity: \textit{On-shell Amplitudes involving a longitudinally polarized gauge boson vanish, provided all other gauge bosons are transversely polarized.} In other words,

\[ \mathcal{M} \equiv e^{\mu_1}e^{\mu_2} \cdots e^{\mu_n} \mathcal{M}_{\mu_1\mu_2 \cdots \mu_n}(\text{momenta}) = 0 \]

when \(e_1^\mu \propto k_1^\mu\) but \(e_2^\nu k_2^\nu = \cdots = e_n^\nu k_n^\nu = 0\).

(c) Verify this identity for the scalar annihilation amplitude.
4. To convert the annihilation amplitude into a cross-section we need to sum / average over the colors of all the particles. As a first step in this direction, it’s convenient to write the amplitude as

$$\mathcal{M}(j + i \rightarrow a + b) = F \times \{T^a, T^b\}^i_j + iG \times [T^a, T^b]^i_j$$  \hspace{1cm} (6)

where $j$ is the ‘color’ index of the scalar particle belonging to some representation $(r)$ of the gauge group $G$, $i$ is the color index of the scalar anti-particle belonging to the conjugate representation $(\bar{r})$, and $a$ and $b$ are the color indices of the gauge bosons belonging to the adjoint representation.

(a) Show that the annihilation amplitude indeed has form (6) and write down the coefficients $F$ and $G$ as explicit functions of the particles momenta and polarizations.

(b) Next, let us sum the $|\mathcal{M}|^2$ over the gauge boson’s colors and average over the scalars’ colors. Show that

$$\frac{1}{\text{dim}^2(r)} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 = \frac{C(r)}{\text{dim}(r)} \times (4C(r) \times |F|^2 + C(\text{adj}) \times (|G|^2 - |F|^2)) \ . \hspace{1cm} (7)$$

In particular, for scalars in the fundamental representation of the $SU(N)$ gauge group,

$$\frac{1}{N^2} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 = \frac{N^2 - 1}{2N^2} \left( \frac{N^2 - 2}{N} |F|^2 + N|G|^2 \right) \ . \hspace{1cm} (8)$$

(c) Evaluate $F$ and $G$ in the center of mass frame. In this frame, the vector particles’ polarizations $e^{\mu}_{1,2} = (0, e_{1,2})$ are purely spatial and transverse to the vectors momenta $\pm \mathbf{k}$. For simplicity, use planar rather than circular polarizations.

(d) Finally, calculate the (polarized, partial) cross-section for the annihilation process.