1. First, a bit of group theory. Consider a generic simple non-abelian Lie group G and its generators T^a. The (quadratic) Casimir operator $C_2 = \sum_a T^a T^a$ commutes with all the generators and hence for any irreducible representation (r) of the group, C_2 restricted to (r) is simply a unit matrix times a number $C(r)$. In other words, if $T^a_{(r)}$ is a matrix of the generator T^a in the representation (r), then $\sum_a T^a_{(r)} T^a_{(r)} = C(r) \times \mathbf{1}$. For example, for the isospin group $SU(2)$, the irreps are characterized by the isospin I and $C(I) = I(I+1)$.

   (a) By symmetry, for any complete representation (r) of the group,

   $$\mathrm{tr}_{(r)}(T^a T^b) \;\equiv\; \mathrm{tr}\left(T^a_{(r)} T^b_{(r)}\right) \;=\; R(r)\delta^{ab} \tag{1}$$

   for some coefficient $R(r)$. Show that for any irreducible representation,

   $$\frac{R(r)}{C(r)} \;=\; \frac{\dim(r)}{\dim(G)}\,. \tag{2}$$

   In particular, for the $SU(2)$ group, this formula gives $R(I) = \frac{1}{3} I(I+1)(2I+1)$.

   (b) Suppose the first three generators of G generate an $SU(2)$ subgroup. Show that if a representation (r) of G decomposes into several $SU(2)$ multiplets of isospins $I_1, I_2, \ldots, I_n$, then

   $$R(r) \;= \sum_{i=1}^{n} \tfrac{1}{3} I_i(I_i+1)(2I_i+1). \tag{3}$$

   (c) Now consider the $SU(N)$ group with an obvious $SU(2)$ subgroup of matrices acting on the first two components of a complex N-vector. The fundamental representation (N) of the $SU(N)$ decomposes into one doublet and $(N-2)$ singlets of the $SU(2)$ subgroup, hence

   $$R(N) \;=\; \tfrac{1}{2} \qquad \text{and} \quad C(N) \;=\; \frac{N^2-1}{2N}\,. \tag{4}$$

   Show that the adjoint representation of the $SU(N)$ decomposes into one $SU(2)$

triplet, $2(N-2)$ doublets and $(N-2)^2$ singlets and hence

$$R(\text{adj}) \;=\; C(\text{adj}) \;\equiv\; C(G) \;=\; N. \tag{5}$$

Hint: $(N) \times (\overline{N}) = (\text{adj}) + (1)$.

(d) The symmetric and the anti-symmetric 2–index tensors form irreducible representations of the $SU(N)$ group. Find out the decomposition of these irreps under an $SU(2) \subset SU(N)$ and calculate their respective R factors.

2. And now let's apply group theory to a physical process of quark-antiquark pair production in *Quantum ChromoDynamics* (QCD). Specifically, let us focus on the $u\bar{u} \to d\bar{d}$ process so there is only one tree-level diagram contributing to this process. Draw this diagram and calculate the amplitude, then sum/average the $|\mathcal{M}|^2$ over both spins and *colors* of the final/initial particles and calculate the total cross section. For simplicity, you may neglect the quark masses.

   Note that the $u\bar{u} \to d\bar{d}$ pair production in QCD is very similar to the $e^-e^+ \to \mu^-\mu^+$ pair production in QED, so the only new aspect of this problem is summing over the colors.

3. Next, consider a scalar analogue of QCD or more generally a theory of Yang–Mills fields A^a_μ and complex scalars Φ_i in some representation (r) of the gauge group G.

   (a) Write down the Lagrangian and the Feynman rules of this theory.

   Next, consider the annihilation process $\Phi + \Phi^* \to 2$ gauge bosons. At the tree level, there are four Feynman diagrams contributing to this process.

   (b) Draw the diagrams and write down the tree-level annihilation amplitude.

   As discussed in class, amplitudes involving the non-abelian gauge fields satisfy a weak form of the Ward identity: *On-shell Amplitudes involving* **a** *longitudinally polarized gauge boson vanish, provided all other gauge bosons are transversely polarized.* In other words,

$$\begin{aligned} \mathcal{M} \;&\equiv\; e_1^{\mu_1} e_2^{\mu_2} \cdots e_n^{\mu_n}\, \mathcal{M}_{\mu_1\mu_2\cdots\mu_n}(\text{momenta}) \;=\; 0 \\ \text{when} \quad e_1^\mu \propto k_1^\mu \quad &\text{but} \quad e_2^\nu k_{2\nu} \;=\; \cdots \;=\; e_n^\nu k_{n\nu} \;=\; 0. \end{aligned}$$

   (c) Verify this identity for the scalar annihilation amplitude.

4. To convert the annihilation amplitude into a cross-section we need to sum / average over the colors of all the particles. As a first step in this direction, it's convenient to write the amplitude as

$$\mathcal{M}(j+i \to a+b) \;=\; F \times \{T^a, T^b\}^i{}_j \;+\; iG \times [T^a, T^b]^i{}_j \tag{6}$$

where j is the 'color' index of the scalar particle belonging to some representation (r) of the gauge group G, i is the color index of the scalar anti-particle belonging to the conjugate representation $(\bar{r})$, and a and b are the color indices of the gauge bosons belonging to the adjoint representation.

(a) Show that the annihilation amplitude indeed has form (6) and write down the coefficients F and G as explicit functions of the particles momenta and polarizations.

(b) Next, let us sum the $|\mathcal{M}|^2$ over the gauge boson's colors and average over the scalars' colors. Show that

$$\frac{1}{\dim^2(r)} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 \;=\; \frac{C(r)}{\dim(r)} \times \left(4C(r) \times |F|^2 \,+\, C(\mathrm{adj}) \times (|G|^2 - |F|^2)\right). \tag{7}$$

In particular, for scalars in the fundamental representation of the $SU(N)$ gauge group,

$$\frac{1}{N^2} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 \;=\; \frac{N^2-1}{2N^2} \left(\frac{N^2-2}{N}\,|F|^2 \,+\, N|G|^2\right). \tag{8}$$

(c) Evaluate F and G in the center of mass frame. In this frame, the vector particles' polarizations $e^{\mu}_{1,2} = (0, \mathbf{e}_{1,2})$ are purely spatial and transverse to the vectors momenta $\pm\mathbf{k}$. For simplicity, use planar rather than circular polarizations.

(d) Finally, calculate the (polarized, partial) cross-section for the annihilation process.