Problem $\mathbf{1}$(a):

Let $\Delta T^{\mu\nu} = \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu}$. Regardless of the specific form of the $\mathcal{K}^{[\lambda\mu]\nu}(\phi, \partial\phi)$ tensor, its anti-symmetry with respect to its first two indices implies

$$\partial_\mu \Delta T^{\mu\nu} \;=\; \partial_\mu \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu} \;=\; 0 \tag{S.1}$$

and hence the first eq. (3). Furthermore,

$$\int d^3\mathbf{x}\,\left(\Delta T^{0\nu} \;=\; \partial_i \mathcal{K}^{i0\nu}\right) \;=\; \oint\limits_{\substack{\text{boundary}\\ \text{of space}}} d^2\text{Area}_{\,i}\, \mathcal{K}^{i0\nu} \;\longrightarrow\; 0 \tag{S.2}$$

when the integral is taken over the whole space, hence the second eq. (3).

Problem $\mathbf{1}$(b):

In the Noether's formula (1) for the stress-energy tensor, ϕ_a stand for the independent fields, however labeled. In the electromagnetic case, the independent fields are components of the 4–vector $A_\lambda(x)$, hence

$$\begin{aligned} T^{\mu\nu}_{\text{Noether}}(\text{EM}) \;&=\; \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\,\partial^\nu A_\lambda \;-\; g^{\mu\nu}\,\mathcal{L} \\ &=\; -F^{\mu\lambda}\,\partial^\nu A_\lambda \;+\; \tfrac{1}{4} g^{\mu\nu}\,F_{\kappa\lambda}F^{\kappa\lambda}. \end{aligned} \tag{S.3}$$

While the second term here is clearly both gauge invariant and symmetric in $\mu \leftrightarrow \nu$, the first term is neither.

Problem $\mathbf{1}$(c):

Clearly, one can easily restore both symmetry and gauge invariance of the electromagnetic stress-energy tensor by replacing $\partial^\nu A_\lambda$ in eq. (S.3) with $F^\nu_{\ \lambda}$, hence eq. (5). The correction amounts to

$$\begin{aligned} \Delta T^{\mu\nu} &= T^{\mu\nu}_{\text{true}} - T^{\mu\nu}_{\text{Noether}} \\ &= -F^{\mu\lambda}\,(F^{\nu}_{\ \lambda} - \partial^\nu A_\lambda = -\partial_\lambda A^\nu)) \\ &= \partial_\lambda \left(F^{\mu\lambda} A^\nu\right) - A^\nu \left(\partial_\lambda F^{\mu\lambda}\right) \end{aligned} \tag{S.4}$$

where the last term on the right hand side vanishes for the free electromagnetic field (which satisfies $\partial_\lambda F^{\mu\lambda} = 0$). Consequently,

$$T^{\mu\nu} = T^{\mu\nu}_{\text{Noether}} + \mathcal{K}^{\lambda\mu\nu} \qquad \text{where} \quad \mathcal{K}^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu = -\mathcal{K}^{\mu\lambda\nu} \tag{S.5}$$

in perfect agreement with eq. (2).

Problem **1**(d):

As explained in class, the Lagrangian $\mathcal{L} = -\frac{1}{4} F_{\kappa\lambda} F^{\kappa\lambda} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)$. Combining this fact with eq. (5), we have the energy density

$$\mathcal{H} = T^{00} = -F^{0i} F^{0}_{\ i} - \mathcal{L} = +\mathbf{E}^2 - \tfrac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) = \tfrac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) \tag{S.6}$$

in agreement with the standard electromagnetic formulæ (note the $c = 1$, *rationalized* units here). Likewise, the energy flux and the momentum density are

$$S^i = T^{i0} = T^{0i} = -F^{0j} F^{i}_{\ j} = -(-E^j)(\epsilon^{ijk} B^k) = +\epsilon^{ijk} E^j B^k = (\mathbf{E} \times \mathbf{B})^i, \tag{S.7}$$

in agreement with the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}$ (again, in the $c = 1$, *rationalized* units). Finally, the (3–dimensional) stress tensor is

$$\begin{aligned} T^{ij}_{\text{EM}} &= -F^{i\lambda} F^{j}_{\ \lambda} - g^{ij} \mathcal{L} = -F^{i0} F^{j}_{\ 0} - F^{ik} F^{j}_{\ k} + \delta^{ij} \mathcal{L} \\ &= -E^i E^j + \epsilon^{ik\ell} B^\ell \, \epsilon^{jkm} B^m + \tfrac{1}{2} \delta^{ij} \left(\mathbf{E}^2 - \mathbf{B}^2\right) \\ &= -E^i E^j - B^i B^j + \tfrac{1}{2} \delta^{ij} \left(\mathbf{E}^2 + \mathbf{B}^2\right). \end{aligned} \tag{S.8}$$

Problem **2**(a):

In a sense, eq. (7) follows from eq. (S.4), but it is just as easy to derive it directly from Maxwell equations. Starting with eq. (5), we immediately have

$$\partial_\mu T^{\mu\nu}_{\rm EM} \;=\; -\bigl(\partial_\mu F^{\mu\lambda}\bigr)F^{\nu}_{\ \lambda} \;-\; F^{\mu\lambda}\bigl(\partial_\mu F^{\nu}_{\ \lambda}\bigr) \;+\; \tfrac{1}{2}F_{\kappa\lambda}\bigl(\partial^\nu F^{\kappa\lambda}\bigr). \tag{S.9}$$

Using the antisymmetry $F^{\mu\lambda}=-F^{\lambda\mu}$, we rewrite the second term on the right hand side as

$$-\,F^{\mu\lambda}\,\partial_\mu F^{\nu}_{\ \lambda} \;=\; +F_{\mu\lambda}\,\partial^\mu F^{\lambda\nu} \;=\; +F_{\mu\lambda}\,\partial^\lambda F^{\nu\mu} \;=\; \tfrac{1}{2}F_{\mu\lambda}\bigl(\partial^\lambda F^{\nu\mu}+\partial^\mu F^{\lambda\nu}\bigr) \;=\; -\tfrac{1}{2}F_{\mu\lambda}\bigl(\partial^\nu F^{\mu\lambda}\bigr) \tag{S.10}$$

where the last equality follows from the homogeneous Maxwell equation $\partial^\lambda F^{\nu\mu}+\partial^\mu F^{\lambda\nu}+\partial^\nu F^{\mu\lambda}=0$. Consequently, the second and the third terms on the right hand side of eq. (S.9) cancel each other and we are left with the first term only. Thus,

$$\partial_\mu T^{\mu\nu}_{\rm EM} \;=\; -\bigl(\partial_\mu F^{\mu\lambda}\bigr)F^{\nu}_{\ \lambda}\,. = \;-\;J^\lambda\,F^{\nu}_{\ \lambda} \tag{S.11}$$

where the second equality comes from the in-homogeneous Maxwell equation $\partial_\mu F^{\mu\lambda}=J^\lambda$. This proves eq. (7), and eq. (8) follows from that and eq. (6). $\mathcal{Q.E.D.}$

Problem **2**(b):

The covariant derivatives in the Lagrangian (9) hide the A_μ dependence:

$$\frac{\partial D_\nu\Phi}{\partial A_\mu} \;=\; iq\Phi\delta^\mu_\nu \quad\text{and}\quad \frac{\partial D_\nu\Phi^*}{\partial A_\mu} \;=\; -iq\Phi^*\delta^\mu_\nu\,, \tag{S.12}$$

and therefore

$$J^\mu \;=\; -\frac{\partial\mathcal{L}}{\partial A_\mu} \;=\; -iq\Phi\,\frac{\partial\mathcal{L}}{\partial D_\mu\Phi} \;+\; -iq\Phi^*\,\frac{\partial\mathcal{L}}{\partial D_\mu\Phi^*} \;=\; -iq\Phi D^\mu\Phi^* \;+\; iq\Phi^*\,D^\mu\Phi. \tag{S.13}$$

Thanks to the covariance of the derivative D^μ this current is gauge invariant:

$$\left.\begin{aligned} \Phi(x)\to e^{+iq\alpha(x)}\Phi(x),\quad D^\mu\Phi(x)\to e^{+iq\alpha(x)}D^\mu\Phi(x),\\ \Phi^*(x)\to e^{-iq\alpha(x)}\Phi^*(x),\quad D^\mu\Phi^*(x)\to e^{-iq\alpha(x)}D^\mu\Phi^*(x) \end{aligned}\right\}\Longrightarrow\; J^\mu(x)\to J^\mu(x). \tag{S.14}$$

$\mathcal{Q.E.D.}$

Problem **2**(c):

As discussed in class, the equations of motions for the scalar fields Φ and Φ^* can be written in gauge invariant form as

$$D_\mu D^\mu \Phi \;=\; -m^2\Phi \quad\text{and}\quad D_\mu D^\mu \Phi^* \;=\; -m^2\Phi^*. \tag{S.15}$$

Consequently,

$$\Phi^*(D_\mu D^\mu\Phi) \;-\; \Phi(D_\mu D^\mu\Phi^*) \;=\; 0, \tag{S.16}$$

which in turn leads to conservation of the electric current (S.13):

$$\begin{aligned} \partial_\mu(\Phi^* D^\mu\Phi) \;&=\; D_\mu(\Phi^* D^\mu\Phi) \;=\; (D_\mu\Phi^*)(D^\mu\Phi) \;+\; \Phi^*\,(D_\mu D^\mu\Phi),\\ \partial_\mu(\Phi D^\mu\Phi^*) \;&=\; D_\mu(\Phi D^\mu\Phi^*) \;=\; (D_\mu\Phi)(D^\mu\Phi^*) \;+\; \Phi\,(D_\mu D^\mu\Phi^*), \end{aligned} \tag{S.17}$$

and therefore

$$\begin{aligned} \partial_\mu J^\mu \;&=\; -iq\partial_\mu(\Phi^* D^\mu\Phi) \;+\; iq\partial_\mu(\Phi D^\mu\Phi^*)\\ &=\; -iq\Phi^*\,D_\mu D^\mu\Phi \;+\; iq\Phi\,D_\mu D^\mu\Phi^* \;=\; 0. \end{aligned} \tag{S.18}$$

Problem **2**(d):

According to the Noether theorem (1),

$$\begin{aligned} T^{\mu\nu}_{\text{Noether}} \;&=\; \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\,\partial^\nu A_\lambda \;+\; \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\,\partial^\nu\Phi \;+\; \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)}\,\partial^\nu\Phi^* \;-\; g^{\mu\nu}\,\mathcal{L}\\ &=\; T^{\mu\nu}_{\text{Noether}}(\text{EM}) \;+\; T^{\mu\nu}_{\text{Noether}}(\text{matter}) \end{aligned} \tag{S.19}$$

where $T^{\mu\nu}_{\text{Noether}}(\text{EM})$ is exactly as in eq. (S.3) for free EM fields and

$$T^{\mu\nu}_{\text{Noether}}(\text{matter}) \;=\; D^\mu\Phi^*\,\partial^\nu\Phi \;+\; D^\mu\Phi\,\partial^\nu\Phi^* \;-\; g^{\mu\nu}\big(D^\lambda\Phi^*\,D_\lambda\Phi \;-\; m^2\Phi^*\Phi\big). \tag{S.20}$$

Both terms on the second line of eq. (S.19) lack $\mu\leftrightarrow\nu$ symmetry and gauge invariance and thus need corrections à la (2). In fact, the same $\mathcal{K}^{[\lambda\mu]\nu} = F^{\mu\lambda}A^\nu$ we used to improve the free electromagnetic stress-energy tensor will now improve both the $T^{\mu\nu}_{\text{EM}}$ and $T^{\mu\nu}_{\text{mat}}$ at the same time!

Indeed, to improve the scalar fields' stress-energy tensor we need

$$\begin{aligned}
\Delta T^{\mu\nu}_{\text{matter}} &\equiv T^{\mu\nu}_{\text{true}}(\text{matter}) \ - \ T^{\mu\nu}_{\text{Noether}}(\text{matter}) \\
&= D^\mu\Phi^*(D^\nu\Phi - \partial^\nu\Phi) \ + \ D^\mu\Phi(D^\nu\Phi^* - \partial^\nu\Phi^*) \\
&= D^\mu\Phi^*(iqA^\nu\Phi) \ + \ D^\mu\Phi(-iqA^\nu\Phi^*) \\
&= -A^\nu(iq\Phi^* D^\mu\Phi \ - \ iq\Phi D^\mu\Phi^*) \\
&= -A^\nu J^\mu,
\end{aligned} \tag{S.21}$$

while the improvement of the EM stress-energy requires (*cf.* eq. (S.4))

$$\Delta T^{\mu\nu}_{\text{EM}} \ = \ \partial_\lambda\left(F^{\mu\lambda}A^\nu\right) \ + \ A^\nu\left(-\partial_\lambda F^{\mu\lambda} \ = \ +J^\mu\right). \tag{S.22}$$

Altogether, to symmetrize the whole stress-energy tensor, we need

$$\Delta T^{\mu\nu}_{\text{tot}} \ \equiv \ T^{\mu\nu}_{\text{true}}(\text{total}) \ - \ T^{\mu\nu}_{\text{Noether}}(\text{total}) \ = \ \partial_\lambda\left(F^{\mu\lambda}A^\nu \ \equiv \ \mathcal{K}^{[\lambda\mu]\nu}\right).$$

$\mathcal{Q.E.D.}$

Problem **2**(e):

Because the fields $\Phi(x)$ and $\Phi^*(x)$ have opposite electric charges, their product is neutral and therefore $\partial_\mu(\Phi^*\Phi) = D_\mu(\Phi^*\Phi) = (D_\mu\Phi^*)\Phi + \Phi^*(D_\mu\Phi)$. Similarly,

$$\begin{aligned}
\partial_\mu\left((D^\mu\Phi^*)\,(D^\nu\Phi)\right) &= (D_\mu D^\mu\Phi^*)\,(D^\nu\Phi) \ + \ (D^\mu\Phi^*)\,(D_\mu D^\nu\Phi) \\
&= -m^2\Phi^*\,(D^\nu\Phi) \ + \ (D_\mu\Phi^*)\,(D^\nu D^\mu\Phi \ + \ iqF^{\mu\nu}\Phi)
\end{aligned} \tag{S.23}$$

where we have applied the field equation $\left(D_\mu D^\mu + m^2\right)\Phi^*(x) = 0$ to the first term on the right hand side and used eq. (14) to expand the second term. Likewise,

$$\begin{aligned}
\partial_\mu\left((D^\mu\Phi)\,(D^\nu\Phi^*)\right) &= (D_\mu D^\mu\Phi)\,(D^\nu\Phi^*) \ + \ (D^\mu\Phi)\,(D_\mu D^\nu\Phi^*) \\
&= -m^2\Phi\,(D^\nu\Phi^*) \ + \ (D_\mu\Phi)\,(D^\nu D^\mu\Phi^* \ - \ iqF^{\mu\nu}\Phi^*)
\end{aligned} \tag{S.24}$$

and

$$\begin{aligned}
\partial_\mu\left[-g^{\mu\nu}\left(D_\lambda\Phi^*\,D^\lambda\Phi \ - \ m^2\,\Phi^*\Phi\right)\right] &= -\partial^\nu\left(D_\lambda\Phi^*\,D^\lambda\Phi\right) \ + \ m^2\partial^\nu\left(\Phi^*\Phi\right) \\
&= -\left(D^\nu D^\mu\Phi^*\right)(D_\mu\Phi) \ - \ (D_\mu\Phi^*)\,(D^\nu D^\mu\Phi) \\
&\qquad + \ m^2\Phi\,(D^\nu\Phi^*) \ + \ m^2\Phi^*\,(D^\nu\Phi)\,.
\end{aligned} \tag{S.25}$$

Together, the left hand sides of eqs. (S.23), (S.24) and (S.25) comprise $\partial_\mu T^{\mu\nu}_{\text{mat}}$ — *cf.* eq. (12).

On the other hand, combining the right hand sides of these three equations results in massive cancellation of all terms except those containing the gauge field strength tensor $F^{\mu\nu}$. Thus,

$$\begin{aligned}
\partial_\mu T^{\mu\nu}_{\text{mat}} &= (D_\mu\Phi^*)\,(iqF^{\mu\nu}\Phi) \;+\; (D_\mu\Phi)\,(-iqF^{\mu\nu}\Phi^*) \\
&= F^{\mu\nu}\,(iq\Phi\,D_\mu\Phi^* \;-\; iq\Phi^*\,D_\mu\Phi) \\
&= F^{\mu\nu}\,J_\nu\,.
\end{aligned} \tag{S.26}$$

$\mathcal{Q}.\mathcal{E}.\mathcal{D}.$