

Problem 1(a):

As discussed in class for the massless case (EM), $\partial\mathcal{L}/\partial(\partial_\mu A_\nu) = -F^{\mu\nu}$.

Clearly, $\partial\mathcal{L}/\partial(A_\nu) = +m^2 A^\nu - J^\nu$. Hence, the Euler–Lagrange field equation is

$$-\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} + \frac{\partial\mathcal{L}}{\partial(A_\nu)} \equiv \partial_\mu F^{\mu\nu} + m^2 A^\nu - J^\nu = 0, \quad (\text{S.1})$$

or in terms of A^ν and their explicit derivatives,

$$\partial^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) + m^2 A^\nu - J^\nu = 0. \quad (\text{S.2})$$

Problem 1(b):

Take the divergence ∂_ν of the field equation (S.2); the first two terms cancel out while the rest becomes

$$m^2 \partial_\nu A^\nu - \partial_\nu J^\nu = 0. \quad (\text{S.3})$$

In the massless case, this equation enforces the current conservation $\partial_\nu J^\nu = 0$ regardless of the 4–vector potential $A^\nu(x)$, but there is no such constraint in the massive case at hand. Instead, eq. (S.3) simply relates the current divergence to the 4–potential divergence. In particular, *if the current happens to satisfy $\partial_\nu J^\nu$, then — and only then — eq. (S.3) requires $\partial_\nu A^\nu = 0$ as well.* Consequently, the field equation (S.2) simplifies to $(\partial^2 + m^2)A^\nu = J^\nu$.
Q.E.D.

Problem 1(c):

As in (a), $\partial\mathcal{L}/\partial(\partial^\mu A^\nu) = -F_{\mu\nu}$.

In particular, $\partial\mathcal{L}/\partial(\partial^0 A^i) = -F_{0i} = -E^i$ while $\partial\mathcal{L}/\partial(\partial^0 A^0) = -F_{00} = 0$.

Problem 1(d):

In terms of the Hamiltonian and Lagrangian densities, eq. (3) means

$$\mathcal{H} = -\dot{\mathbf{A}} \cdot \mathbf{E} - \mathcal{L}. \quad (3')$$

In terms of \mathbf{A} , \mathbf{E} and A_0 ,

$$\begin{aligned} \dot{\mathbf{A}} &= -\mathbf{E} - \nabla A_0, \\ -\dot{\mathbf{A}} \cdot \mathbf{E} &= \mathbf{E}^2 + \mathbf{E} \cdot \nabla A_0, \\ \mathcal{L} &= \frac{1}{2} (\mathbf{E}^2 - (\nabla \times \mathbf{A})^2) + \frac{1}{2} m^2 (A_0^2 - \mathbf{A}^2) - (A_0 J_0 - \mathbf{A} \cdot \mathbf{J}). \end{aligned} \quad (S.4)$$

Consequently,

$$\mathcal{H} = \frac{1}{2} \mathbf{E}^2 + \mathbf{E} \cdot \nabla A_0 - \frac{1}{2} m^2 A_0^2 + A_0 J_0 + \frac{1}{2} (\nabla \times \mathbf{A})^2 + \frac{1}{2} m^2 \mathbf{A}^2 - \mathbf{A} \cdot \mathbf{J}, \quad (S.5)$$

which immediately leads to eq. (4) (after one integration by parts). *Q.E.D.*

Problem 1(e):

Expanding the left hand side of eq. (5) gives us

$$\frac{\delta H}{\delta A_0(\mathbf{x})} \equiv \frac{\partial \mathcal{H}}{\partial (A_0)} - \nabla_i \frac{\partial \mathcal{H}}{\partial (\nabla_i A_0)} = -m^2 A_0 + J_0 - \nabla_i E^i,$$

hence

$$A_0(\mathbf{x}, t) = \frac{J_0 - \nabla \cdot \mathbf{E}}{m^2} \quad (S.6)$$

everywhere in spacetime. Please note that there are no derivatives at the left hand side of this formula.

Likewise, expanding the right hand side of the first eq. (6) yields

$$\frac{\delta H}{\delta E^i(\mathbf{x})} \equiv \frac{\partial \mathcal{H}}{\partial(E^i)} - \nabla_j \frac{\partial \mathcal{H}}{\partial(\nabla_j E^i)} = E^i + \nabla_i A_0$$

and hence

$$\frac{\partial}{\partial t} \mathbf{A}(\mathbf{x}, t) = -\mathbf{E} - \nabla A_0. \quad (\text{S.7})$$

Similarly,

$$\frac{\delta H}{\delta A^i(\mathbf{x})} \equiv \frac{\partial \mathcal{H}}{\partial(A^i)} - \nabla_j \frac{\partial \mathcal{H}}{\partial(\nabla_j A^i)} = m^2 A^i - J^i - \nabla_j (\epsilon^{jik} (\nabla \times \mathbf{A})^k)$$

and hence

$$\frac{\partial}{\partial t} \mathbf{E}(\mathbf{x}, t) = m^2 \mathbf{A} - \mathbf{J} + \nabla \times (\nabla \times \mathbf{A}). \quad (\text{S.8})$$

Problem 1(f):

In the 3-space notations, the Euler–Lagrange field equations (S.1) become

$$\nabla \cdot \mathbf{E} - m^2 A_0 = J_0, \quad (\text{S.9})$$

$$-\dot{\mathbf{E}} + \nabla \times \mathbf{B} + m^2 \mathbf{A} = \mathbf{J}, \quad (\text{S.10})$$

where

$$\mathbf{E} \stackrel{\text{def}}{=} -\dot{\mathbf{A}} - \nabla A_0, \quad (\text{S.11})$$

$$\mathbf{B} \stackrel{\text{def}}{=} \nabla \times \mathbf{A}. \quad (\text{S.12})$$

Clearly, eq. (S.9) is equivalent to eq. (S.6) while eq. (S.10) is equivalent to eq. (S.8) (provided \mathbf{B} is defined as in eq. (S.12)). Finally, eq. (S.11) is equivalent to eq. (S.7), although their origins differ: In the Lagrangian formalism, eq. (S.11) is the definition of the \mathbf{E} field in terms of A_0 , \mathbf{A} and their derivatives, while in the Hamiltonian formalism, \mathbf{E} is an independent conjugate field and eq. (S.7) is the dynamical equation of motion for the $\dot{\mathbf{A}}$. *Q.E.D.*

Problem 2(a):

According to the time-independent Maxwell equations (7), the divergences $\nabla \cdot \hat{\mathbf{E}}$ and $\nabla \cdot \hat{\mathbf{B}}$ vanish as operators and therefore should commute with any other operator in the theory. On the other hand, the commutation relations of the $\nabla \cdot \hat{\mathbf{E}}$ and $\nabla \cdot \hat{\mathbf{B}}$ follow directly from those of the $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$ field themselves:

$$\begin{aligned} [\nabla \cdot \hat{\mathbf{E}}(\mathbf{x}), \text{whatever}(\mathbf{x}')] &= \frac{\partial}{\partial x^i} [\hat{E}^i(\mathbf{x}), \text{same whatever}(\mathbf{x}')], \\ [\nabla \cdot \hat{\mathbf{B}}(\mathbf{x}), \text{whatever}(\mathbf{x}')] &= \frac{\partial}{\partial x^i} [\hat{B}^i(\mathbf{x}), \text{same whatever}(\mathbf{x}')]. \end{aligned} \quad (\text{S.13})$$

Thus to make sure that the fields' commutation relations (8) are consistent with eqs. (7), we must verify that

$$\begin{aligned} \frac{\partial}{\partial x^i} [\hat{E}^i(\mathbf{x}), \hat{E}^j(\mathbf{x}')] &= 0, \\ \frac{\partial}{\partial x^i} [\hat{E}^i(\mathbf{x}), \hat{B}^j(\mathbf{x}')] &= 0, \\ \frac{\partial}{\partial x^i} [\hat{B}^i(\mathbf{x}), \hat{E}^j(\mathbf{x}')] &= 0, \\ \frac{\partial}{\partial x^i} [\hat{B}^i(\mathbf{x}), \hat{B}^j(\mathbf{x}')] &= 0, \end{aligned} \quad (\text{S.14})$$

Plugging in the commutation relations (8) into these formulæ, we immediately see that

$$\frac{\partial}{\partial x^i} [\hat{E}^i(\mathbf{x}), \hat{E}^j(\mathbf{x}')] = \frac{\partial}{\partial x^i} [\hat{B}^i(\mathbf{x}), \hat{B}^j(\mathbf{x}')] = 0 \quad (\text{S.15})$$

while

$$\frac{\partial}{\partial x^i} [\hat{E}^i(\mathbf{x}), \hat{B}^j(\mathbf{x}')] = \frac{\partial}{\partial x^i} \left(-i\hbar c \epsilon^{ijk} \frac{\partial}{\partial x^k} \delta^{(3)}(\mathbf{x} - \mathbf{x}') \right) = 0 \quad (\text{S.16})$$

because $\epsilon^{ijk} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^k} \equiv 0$. Likewise,

$$\begin{aligned} \frac{\partial}{\partial x^i} [\hat{B}^i(\mathbf{x}), \hat{E}^j(\mathbf{x}')] &= \frac{\partial}{\partial x^i} \left(+i\hbar c \epsilon^{jik} \frac{\partial}{\partial x'^k} \delta^{(3)}(\mathbf{x} - \mathbf{x}') \right) \\ &= -i\hbar c \epsilon^{jik} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^k} \delta^{(3)}(\mathbf{x} - \mathbf{x}') = 0. \end{aligned} \quad (\text{S.17})$$

Q.E.D.

Problem 2(b):

According to eqs. (8), at equal times $[\hat{B}^i, \frac{1}{2}\hat{\mathbf{B}}^2(\mathbf{x}')] = 0$ while

$$[\hat{B}^i, \frac{1}{2}\hat{\mathbf{E}}^2(\mathbf{x}')] = \hat{E}^j(\mathbf{x}') \times [\hat{B}^i(\mathbf{x}), \hat{E}^j(\mathbf{x}')] = \hat{E}^j(\mathbf{x}') \times (+i\hbar c)\epsilon^{jik} \frac{\partial}{\partial x'^k} \delta^{(3)}(\mathbf{x}' - \mathbf{x}), \quad (\text{S.18})$$

hence

$$[\hat{B}^i(\mathbf{x}), \hat{H}] = \int d^3\mathbf{x}' (+i\hbar c)\epsilon^{jik} \hat{E}^j(\mathbf{x}') \times \frac{\partial}{\partial x'^k} \delta^{(3)}(\mathbf{x}' - \mathbf{x}) = -i\hbar c \epsilon^{jik} \nabla_k \hat{E}^j(\mathbf{x}), \quad (\text{S.19})$$

or in vector notations,

$$[\hat{\mathbf{B}}(\mathbf{x}), \hat{H}] = -i\hbar c \nabla \times \hat{\mathbf{E}}. \quad (\text{S.20})$$

Consequently, in the Heisenberg picture

$$\frac{1}{c} \frac{\partial}{\partial t} \hat{\mathbf{B}}(\mathbf{x}, t) = -\nabla \times \hat{\mathbf{E}}(\mathbf{x}, t). \quad (\text{9.a})$$

Likewise, $[\hat{E}^i, \frac{1}{2}\hat{\mathbf{E}}^2(\mathbf{x}')] = 0$ while

$$[\hat{E}^i, \frac{1}{2}\hat{\mathbf{B}}^2(\mathbf{x}')] = \hat{B}^j(\mathbf{x}') \times [\hat{E}^i(\mathbf{x}), \hat{B}^j(\mathbf{x}')] = \hat{B}^j(\mathbf{x}') \times (-i\hbar c)\epsilon^{ijk} \frac{\partial}{\partial x^k} \delta^{(3)}(\mathbf{x}' - \mathbf{x}), \quad (\text{S.21})$$

hence

$$[\hat{E}^i(\mathbf{x}), \hat{H}] = \int d^3\mathbf{x}' (-i\hbar c)\epsilon^{ijk} \hat{B}^j(\mathbf{x}') \times \frac{\partial}{\partial x^k} \delta^{(3)}(\mathbf{x}' - \mathbf{x}) = -i\hbar c \epsilon^{ijk} \nabla_k \hat{B}^j(\mathbf{x}), \quad (\text{S.22})$$

or in vector notations,

$$[\hat{\mathbf{E}}(\mathbf{x}), \hat{H}] = +i\hbar c \nabla \times \hat{\mathbf{B}}. \quad (\text{S.23})$$

Consequently, in the Heisenberg picture

$$\frac{1}{c} \frac{\partial}{\partial t} \hat{\mathbf{E}}(\mathbf{x}, t) = +\nabla \times \hat{\mathbf{B}}(\mathbf{x}, t). \quad (\text{9.b})$$

Q.E.D.

Problem 3(a):

In the Hamiltonian formalism for the classical fields $\Phi(x)$ and $\Phi^*(x)$, the canonical conjugate fields are

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Phi^*)} = \partial_0 \Phi(x) \quad \text{and} \quad \Pi^*(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Phi)} = \partial_0 \Phi^*(x). \quad (\text{S.24})$$

The canonical conjugation implies canonical Poisson brackets between the classical fields $\Phi(\mathbf{x})$ and $\Pi^*(\mathbf{x})$, and likewise $\Phi^*(\mathbf{x})$ and $\Pi(\mathbf{x})$ and hence the canonical commutation relation between their quantum counterparts: In the Schrödinger picture

$$\begin{aligned} [\hat{\Phi}(\mathbf{x}), \hat{\Phi}(\mathbf{x}')] &= [\hat{\Phi}(\mathbf{x}), \hat{\Phi}^\dagger(\mathbf{x}')] = [\hat{\Phi}^\dagger(\mathbf{x}), \hat{\Phi}^\dagger(\mathbf{x}')] = 0, \\ [\hat{\Pi}(\mathbf{x}), \hat{\Pi}(\mathbf{x}')] &= [\hat{\Pi}(\mathbf{x}), \hat{\Pi}^\dagger(\mathbf{x}')] = [\hat{\Pi}^\dagger(\mathbf{x}), \hat{\Pi}^\dagger(\mathbf{x}')] = 0, \\ [\hat{\Phi}(\mathbf{x}), \hat{\Pi}(\mathbf{x}')] &= [\hat{\Phi}^\dagger(\mathbf{x}), \hat{\Pi}^\dagger(\mathbf{x}')] = 0, \\ [\hat{\Phi}(\mathbf{x}), \hat{\Pi}^\dagger(\mathbf{x}')] &= [\hat{\Phi}^\dagger(\mathbf{x}), \hat{\Pi}(\mathbf{x}')] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}'). \end{aligned} \quad (\text{S.25})$$

In the Heisenberg picture, we have similar commutation relations for equal times $t = t'$; for un-equal times, the formulæ are much more complicated.

Classically, the Hamiltonian density is

$$\begin{aligned} \mathcal{H} &= \Pi \partial_0 \Phi + \Pi^* \partial_0 \Phi^* - \mathcal{L} \\ &= \Pi^* \Pi + \nabla \Phi^* \cdot \nabla \Phi + m^2 \Phi^* \Phi, \end{aligned} \quad (\text{S.26})$$

so the quantum theory's Hamiltonian is obviously as in eq. (13). *Q.E.D.*

Problem 3(b):

Fourier transforming the canonical commutation relations (S.25) results in

$$\begin{aligned} [\hat{\Phi}_{\mathbf{p}}, \hat{\Phi}_{\mathbf{p}'}] &= [\hat{\Phi}_{\mathbf{p}}, \hat{\Phi}_{\mathbf{p}'}^\dagger] = [\hat{\Phi}_{\mathbf{p}}^\dagger, \hat{\Phi}_{\mathbf{p}'}^\dagger] = 0, \\ [\hat{\Pi}_{\mathbf{p}}, \hat{\Pi}_{\mathbf{p}'}] &= [\hat{\Pi}_{\mathbf{p}}, \hat{\Pi}_{\mathbf{p}'}^\dagger] = [\hat{\Pi}_{\mathbf{p}}^\dagger, \hat{\Pi}_{\mathbf{p}'}^\dagger] = 0, \\ [\hat{\Phi}_{\mathbf{p}}, \hat{\Pi}_{\mathbf{p}'}] &= [\hat{\Phi}_{\mathbf{p}}^\dagger, \hat{\Pi}_{\mathbf{p}'}^\dagger] = 0, \\ [\hat{\Phi}_{\mathbf{p}}, \hat{\Pi}_{\mathbf{p}'}^\dagger] &= [\hat{\Phi}_{\mathbf{p}}^\dagger, \hat{\Pi}_{\mathbf{p}'}] \delta_{\mathbf{p}, \mathbf{p}'}. \end{aligned} \quad (\text{S.27})$$

Consequently,

$$[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{p}'}] = [\hat{a}_{\mathbf{p}}, \hat{b}_{\mathbf{p}'}^\dagger] = [\hat{b}_{\mathbf{p}}^\dagger, \hat{b}_{\mathbf{p}'}^\dagger] = 0 \quad (\text{S.28})$$

because all the $\hat{\Phi}_{\mathbf{p}}$ and all the $\hat{\Pi}_{\mathbf{p}}$ operators commute with each other, and likewise Similarly,

$$[\hat{b}_{\mathbf{p}}, \hat{b}_{\mathbf{p}'}] = [\hat{b}_{\mathbf{p}}, \hat{a}_{\mathbf{p}'}^\dagger] = [\hat{a}_{\mathbf{p}}^\dagger, \hat{a}_{\mathbf{p}'}^\dagger] = 0 \quad (\text{S.29})$$

because all the $\hat{\Phi}_{\mathbf{p}}^\dagger$ and all $\hat{\Pi}_{\mathbf{p}}^\dagger$ operators commute with each other too. Less obviously

$$\begin{aligned} [\hat{a}_{\mathbf{p}}, \hat{b}_{\mathbf{p}'}] &= \frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{p}'}}} (E_{\mathbf{p}}E_{\mathbf{p}'}(0) + iE_{\mathbf{p}}(i\delta_{\mathbf{p},-\mathbf{p}'} + iE_{\mathbf{p}'}(-i\delta_{\mathbf{p},-\mathbf{p}'}) - (0)) \\ &= \delta_{\mathbf{p},-\mathbf{p}'} \times \frac{E_{\mathbf{p}'} - E_{\mathbf{p}}}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{p}'}}} = 0, \end{aligned} \quad (\text{S.30})$$

and similarly $[\hat{a}_{\mathbf{p}}^\dagger, \hat{b}_{\mathbf{p}'}^\dagger] = 0$. Finally,

$$\begin{aligned} [\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{p}'}^\dagger] &= [\hat{b}_{\mathbf{p}}, \hat{b}_{\mathbf{p}'}^\dagger] = \frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{p}'}}} (E_{\mathbf{p}}E_{\mathbf{p}'}(0) - iE_{\mathbf{p}}(i\delta_{\mathbf{p},\mathbf{p}'} + iE_{\mathbf{p}'}(-i\delta_{\mathbf{p},\mathbf{p}'}) + (0)) \\ &= \delta_{\mathbf{p},\mathbf{p}'} \times \frac{E_{\mathbf{p}'} + E_{\mathbf{p}}}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{p}'}}} \\ &= \delta_{\mathbf{p},\mathbf{p}'} . \end{aligned} \quad (\text{S.31})$$

Q.E.D.

Problem 3(c):

First, Fourier-transforming the free Hamiltonian (13) gives us

$$\hat{H}_{\text{free}} = \sum_{\mathbf{p}} \left(\hat{\Pi}_{\mathbf{p}}^\dagger \hat{\Pi}_{\mathbf{p}} + E_{\mathbf{p}}^2 \hat{\Phi}_{\mathbf{p}}^\dagger \hat{\Phi}_{\mathbf{p}} \right). \quad (\text{S.32})$$

Second, we reverse the definitions (15) to obtain

$$\hat{\Phi}_{\mathbf{p}} = \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(\hat{a}_{\mathbf{p}} + \hat{b}_{-\mathbf{p}}^\dagger \right) \quad \text{and} \quad \hat{\Pi}_{\mathbf{p}} = \frac{\sqrt{2E_{\mathbf{p}}}}{2i} \left(\hat{b}_{\mathbf{p}} - \hat{a}_{-\mathbf{p}}^\dagger \right). \quad (\text{S.33})$$

Third, we calculate

$$\begin{aligned}
E_{\mathbf{p}}^2 \hat{\Phi}_{\mathbf{p}}^\dagger \hat{\Phi}_{\mathbf{p}} + \hat{\Pi}_{-\mathbf{p}} \hat{\Pi}_{-\mathbf{p}}^\dagger &= \frac{E_{\mathbf{p}}}{2} (\hat{a}_{\mathbf{p}}^\dagger + \hat{b}_{-\mathbf{p}}) (\hat{a}_{\mathbf{p}} + \hat{b}_{-\mathbf{p}}^\dagger) + \frac{E_{\mathbf{p}}}{2} (\hat{a}_{\mathbf{p}}^\dagger - \hat{b}_{-\mathbf{p}}) (\hat{a}_{\mathbf{p}} - \hat{b}_{-\mathbf{p}}^\dagger) \\
&= E_{\mathbf{p}} \times (\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \hat{b}_{-\mathbf{p}} \hat{b}_{-\mathbf{p}}^\dagger) \\
&= E_{\mathbf{p}} \times (\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \hat{b}_{-\mathbf{p}}^\dagger \hat{b}_{-\mathbf{p}} + 1)
\end{aligned} \tag{S.34}$$

Finally, we put eqs. (S.34) and (S.32) together and derive

$$\begin{aligned}
\hat{H}_{\text{free}} &= \sum_{\mathbf{p}} \left(E_{\mathbf{p}}^2 \hat{\Phi}_{\mathbf{p}}^\dagger \hat{\Phi}_{\mathbf{p}} + \hat{\Pi}_{-\mathbf{p}} \hat{\Pi}_{-\mathbf{p}}^\dagger \right) \\
&= \sum_{\mathbf{p}} E_{\mathbf{p}} (\hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \hat{b}_{-\mathbf{p}}^\dagger \hat{b}_{-\mathbf{p}} + 1) \\
&= \sum_{\mathbf{p}} \left(E_{\mathbf{p}} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + E_{\mathbf{p}} \hat{b}_{-\mathbf{p}}^\dagger \hat{b}_{-\mathbf{p}} \right) + \text{const.}
\end{aligned} \tag{S.35}$$

Q.E.D.

Problem 3(d):

In the Hamiltonian formalism, the classical charge density is

$$J^0 = i\Phi^*(\partial^0\Phi) - i(\partial^0\Phi^*)\Phi = i\Phi^*\Pi - i\Phi\Pi^*, \tag{S.36}$$

(*cf.* eq. (11)). In a quantum theory, this gives us the charge operator

$$\hat{Q} = \int d^3\mathbf{x} \hat{J}^0(\mathbf{x}) = \int d^3\mathbf{x} \left(\frac{i}{2} \{ \hat{\Phi}^\dagger, \hat{\Pi} \} - \frac{i}{2} \{ \hat{\Phi}, \hat{\Pi}^\dagger \} + \text{const} \right), \tag{S.37}$$

where the unknown constant reflects the ambiguity of operator ordering in quantizing a classical quantity. We shall see momentarily that setting this constant to zero results in the normal ordering of \hat{Q} in terms of creation and annihilation operators and hence in zero charge of the vacuum state $|0\rangle$. Physically, this is what we want — hence, the first eq. (17).

Our next step is to Fourier transforming eq. (S.37), which yields

$$\hat{Q} = \sum_{\mathbf{p}} \left(\frac{i}{2} \{ \hat{\Phi}_{\mathbf{p}}^{\dagger}, \hat{\Pi}_{\mathbf{p}} \} - \frac{i}{2} \{ \hat{\Phi}_{\mathbf{p}}, \hat{\Pi}_{\mathbf{p}}^{\dagger} \} \right). \quad (\text{S.38})$$

Next, we re-express the anticommutators here in terms of the creation and annihilator operators according to eqs. (S.33), thus after simple algebra

$$\begin{aligned} \frac{i}{2} \{ \hat{\Phi}_{\mathbf{p}}^{\dagger}, \hat{\Pi}_{\mathbf{p}} \} &= \frac{\hat{a}_{\mathbf{p}}^{\dagger} \hat{a}_{\mathbf{p}} - \hat{b}_{-\mathbf{p}}^{\dagger} \hat{b}_{-\mathbf{p}} + \hat{a}_{\mathbf{p}}^{\dagger} \hat{b}_{-\mathbf{p}}^{\dagger} - \hat{a}_{\mathbf{p}} \hat{b}_{-\mathbf{p}}}{4E_{\mathbf{p}}}, \\ \frac{-i}{2} \{ \hat{\Phi}_{\mathbf{p}}, \hat{\Pi}_{\mathbf{p}}^{\dagger} \} &= \frac{\hat{a}_{\mathbf{p}}^{\dagger} \hat{a}_{\mathbf{p}} - \hat{b}_{-\mathbf{p}}^{\dagger} \hat{b}_{-\mathbf{p}} - \hat{a}_{\mathbf{p}}^{\dagger} \hat{b}_{-\mathbf{p}}^{\dagger} + \hat{a}_{\mathbf{p}} \hat{b}_{-\mathbf{p}}}{4E_{\mathbf{p}}}, \end{aligned} \quad (\text{S.39})$$

and therefore

$$\hat{Q} = \sum_{\mathbf{p}} \left(\hat{a}_{\mathbf{p}}^{\dagger} \hat{a}_{\mathbf{p}} - \hat{b}_{-\mathbf{p}}^{\dagger} \hat{b}_{-\mathbf{p}} \right) = \sum_{\mathbf{p}} \left(\hat{a}_{\mathbf{p}}^{\dagger} \hat{a}_{\mathbf{p}} - \hat{b}_{\mathbf{p}}^{\dagger} \hat{b}_{\mathbf{p}} \right). \quad \mathcal{Q.E.D.}$$

Problem 3(e):

According to eq. (18), classically

$$T^{0i} = \partial^0 \Phi^* \partial^i \Phi + \partial^0 \Phi \partial^i \Phi^* = -\Pi^* \partial_i \Phi - \Pi \partial_i \Phi^*, \quad (\text{S.40})$$

and hence in the quantum theory

$$\hat{\mathbf{P}}_{\text{mech}} = \int d^3 \mathbf{x} \left(-\frac{1}{2} \{ \hat{\Pi}^{\dagger}, \nabla \hat{\Phi} \} - \frac{1}{2} \{ \hat{\Pi}, \nabla \hat{\Phi}^{\dagger} \} \right). \quad (\text{S.41})$$

Fourier-transforming this formula, we arrive at

$$\hat{\mathbf{P}}_{\text{mech}} = \sum_{\mathbf{p}} \left(-\frac{i\mathbf{p}}{2} \{ \hat{\Pi}_{\mathbf{p}}^{\dagger}, \hat{\Phi}_{\mathbf{p}} \} + \frac{i\mathbf{p}}{2} \{ \hat{\Pi}_{\mathbf{p}}, \hat{\Phi}_{\mathbf{p}}^{\dagger} \} \right), \quad (\text{S.42})$$

and hence in light of eqs. (S.39)

$$\hat{\mathbf{P}}_{\text{mech}} = \sum_{\mathbf{p}} \left(\mathbf{p} \hat{a}_{\mathbf{p}}^{\dagger} \hat{a}_{\mathbf{p}} - \mathbf{p} \hat{b}_{-\mathbf{p}}^{\dagger} \hat{b}_{-\mathbf{p}} \right) = \sum_{\mathbf{p}} \left(\mathbf{p} \hat{a}_{\mathbf{p}}^{\dagger} \hat{a}_{\mathbf{p}} + \mathbf{p} \hat{b}_{\mathbf{p}}^{\dagger} \hat{b}_{\mathbf{p}} \right). \quad \mathcal{Q.E.D.}$$