Problem $\mathbf{1}$(a):

First, let us verify eq. (4) for a wave function $\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ of the form (3), that is, for the quantum state $|N,\Psi\rangle = |\alpha_1,\ldots,\alpha_N\rangle = \big|\{n_\beta\}\big\rangle$. Let us define $\Psi'(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})$ according to eq. (3). Using orthonormality of the 1-particle wave functions $\phi_\beta(\mathbf{x})$, we have

$$\begin{aligned}\Psi'(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) &= \frac{\sqrt{N}}{\sqrt{C_{\alpha_1,\ldots,\alpha_N}}}\int d^3\mathbf{x}_N\,\varphi^*_\gamma(\mathbf{x}_N)\times\sum_{(\tilde\alpha_1,\ldots,\tilde\alpha_N)}\varphi_{\tilde\alpha_1}(\mathbf{x}_1)\times\cdots\times\varphi_{\tilde\alpha_N}(\mathbf{x}_N)\\ &= \frac{\sqrt{N}}{\sqrt{C_{\alpha_1,\ldots,\alpha_N}}}\sum_{(\tilde\alpha_1,\ldots,\tilde\alpha_{N-1},\tilde\alpha_N)}\varphi_{\tilde\alpha_1}(\mathbf{x}_1)\times\cdots\times\varphi_{\tilde\alpha_{N-1}}(\mathbf{x}_{N-1})\times\delta_{\tilde\alpha_N,\gamma}\,, \end{aligned}\tag{S.1}$$

which leads to two distinct situations according to the n_γ:

(A) For $n_\gamma=0$ none of the $\alpha_1,\ldots,\alpha_N$ equals γ, hence for any permutation $(\tilde\alpha_1,\ldots,\tilde\alpha_{N-1},\tilde\alpha_N)$ we have $\tilde\alpha_n\neq\gamma$. Consequently, every term in the sum (S.1) vanishes and therefore $\Psi'\equiv 0$. At the same time, $\hat a_\gamma\,\big|\{n_\beta\}\big\rangle = 0$ for $n_\gamma=0$, and therefore

$$\big|(N-1),\Psi'\big\rangle \;=\; 0 \;=\; \hat a_\gamma\,\big|\{n_\beta\}\big\rangle\,. \tag{S.2}$$

(B) On the other hand, for $n_\gamma>0$ we have $\big|\{n_\beta\}\big\rangle = \big|\alpha_1,\ldots,\alpha_{N-1},\gamma\big\rangle$, and permutations with $\tilde\alpha_N=\alpha_N=\gamma$ do contribute to the sum (S.1). Note that we sum over *distinct* permutations of $(\alpha_1,\ldots,\alpha_N)$ only, so even if some of the $\alpha_1,\ldots,\alpha_{N-1}$ are also equal to γ, restricting the sum to permutations $(\tilde\alpha_1,\ldots,\tilde\alpha_{N-1},\tilde\alpha_N)$ with $\tilde\alpha_N=\gamma$ is equivalent to fixing $\tilde\alpha_N=\alpha_N$ and summing over the permutations $(\tilde\alpha_1,\ldots,\tilde\alpha_{N-1})$ of the $(\alpha_1,\ldots,\alpha_{N-1})$. Thus,

$$\Psi'(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \;=\; \frac{\sqrt{N}}{\sqrt{C_{\alpha_1,\ldots,\alpha_N}}}\sum_{\substack{\text{distinct permutations}\\ (\tilde\alpha_1,\ldots,\tilde\alpha_{N-1})\text{ of }(\alpha_1,\ldots,\alpha_{N-1})}}\varphi_{\tilde\alpha_1}(\mathbf{x}_1)\cdots\varphi_{\tilde\alpha_{N-1}}(\mathbf{x}_{N-1}). \tag{S.3}$$

Comparing this expression to eq. (3) for $N'=N-1$, we immediately see that except for the overall normalization, $\Psi'(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})$ is the wave function of the $(N-1)$ particle state

$\big|\alpha_1,\ldots,\alpha_{N-1}\big\rangle$. Specifically,

$$\begin{aligned}
\big|(N-1),\Psi'\big\rangle \;&=\; \frac{\sqrt{N}}{\sqrt{C_{\alpha_1,\ldots,\alpha_{N-1},\alpha_N}}} \times \sqrt{C_{\alpha_1,\ldots,\alpha_{N-1}}} \times \big|\alpha_1,\ldots,\alpha_{N-1}\big\rangle \\
&=\; \sqrt{\frac{N}{n_\gamma} \times \frac{C_{\alpha_1,\ldots,\alpha_{N-1}}}{C_{\alpha_1,\ldots,\alpha_{N-1},\alpha_N}}} \times \hat{a}_\gamma \big|\{n_\beta\}\big\rangle\,.
\end{aligned} \tag{S.4}$$

To fix the normalization factor, remember that $C_{\alpha_1,\ldots,\alpha_N}$ is the number of distinct permutations of the $(\alpha_1,\ldots,\alpha_N)$, which is equal to the number of all permutations divided by the number of trivial permutations of identical α's. In terms of the occupation numbers,

$$C_{\alpha_1,\ldots,\alpha_N} \;=\; \frac{N!}{\prod_\beta n_\beta!}\,. \tag{S.5}$$

Likewise,

$$C_{\alpha_1,\ldots,\alpha_{N-1}} \;=\; \frac{(N-1)!}{\prod_\beta n'_\beta!} \tag{S.6}$$

where $n'_\beta = n_\beta - \delta_{\beta,\gamma}$. Consequently,

$$\frac{C_{\alpha_1,\ldots,\alpha_{N-1}}}{C_{\alpha_1,\ldots,\alpha_{N-1},\alpha_N}} \;=\; \frac{(N-1)!}{N!} \times \frac{n_\gamma!}{(n_\gamma-1)!} \;=\; \frac{n_\gamma}{N}\,, \tag{S.7}$$

and therefore in eq. (S.4) the numerical factors cancel out and

$$\big|(N-1),\Psi'\big\rangle \;=\; \hat{a}_\gamma \big|\{n_\beta\}\big\rangle\,. \tag{S.8}$$

Altogether, combining eqs. (S.2) and (S.8), we find that

$$\forall\,|\Psi\rangle = |\alpha_1,\ldots,\alpha_N\rangle:\;\; \big|\Psi'\big\rangle \;=\; \hat{a}_\gamma\,|\Psi\rangle\,. \tag{S.9}$$

By linearity of eq. (4) it follows that $|\Psi'\rangle = \hat{a}_\gamma\,|\Psi\rangle$ for any linear combination of the $|\alpha_1,\ldots,\alpha_N\rangle$ states. As we saw in class, such states form a complete basis of the N-boson Hilbert space, therefore

$$\left|\Psi'\right\rangle \;=\; \hat{a}_\gamma\left|\Psi\right\rangle \quad \forall\left|\Psi\right\rangle \in \mathcal{H}_{N\,\text{bosons}}\,. \qquad\qquad \mathcal{Q.E.D.}$$

Problem $\mathbf{1}$(b):

We can prove eq. (5) directly from eqs. (1) and (3), but it is easier to use eq. (4) and the fact that the creation operators are hermitian conjugate to the annihilation operators. Thus, for any N particle state $|N,\Psi\rangle$ and any $(N+1)$ particle state $\left|(N+1),\widetilde{\Psi}\right\rangle$,

$$\left\langle (N+1),\widetilde{\Psi}\right|\hat{a}^\dagger_\gamma\left|N,\Psi\right\rangle \;=\; \left\langle N,\Psi\right|\hat{a}_\gamma\left|(N+1),\widetilde{\Psi}\right\rangle^*, \tag{S.10}$$

or in wave-function terms,

$$\begin{aligned}
\int\!\cdots\!\int d^3\mathbf{x}_1\cdots\mathbf{x}_{N+1}\,\widetilde{\Psi}^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N+1})\times\Psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N+1})\\
&= \int\!\cdots\!\int d^3\mathbf{x}_1\cdots\mathbf{x}_N\,\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)\times\widetilde{\Psi}'^*(\mathbf{x}_1,\ldots,\mathbf{x}_N)\\
\langle\!\langle\text{according to eq. (4)}\rangle\!\rangle\quad
&= \int\!\cdots\!\int d^3\mathbf{x}_1\cdots\mathbf{x}_N\,\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)\times\\
&\qquad\times\sqrt{N+1}\int d^3\mathbf{x}_{N+1}\varphi_\gamma(\mathbf{x}_{N+1})\times\widetilde{\Psi}^*(\mathbf{x}_1,\ldots,\mathbf{x}_N,\mathbf{x}_{N+1})\\
&= \int\!\cdots\!\int d^3\mathbf{x}_1\cdots\mathbf{x}_{N+1}\,\widetilde{\Psi}^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N+1})\times\\
&\qquad\times\left[\sqrt{N+1}\,\varphi_\gamma(\mathbf{x}_{N+1})\,\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)\right]
\end{aligned} \tag{S.11}$$

If this equality were to hold true for any $(N+1)$ wave function $\widetilde{\Psi}(\mathbf{x}_1,\ldots,\mathbf{x}_{N+1})$, it would imply

$$\Psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N+1}) \;=\; \sqrt{N+1}\,\varphi_\gamma(\mathbf{x}_{N+1})\,\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N). \tag{S.12}$$

However, we have proved eq. (S.11) only for the bosonic wave functions, thus only for the totally-symmetric $\widetilde{\Psi}(\mathbf{x}_1,\ldots,\mathbf{x}_{N+1})$. Consequently, instead of eq. (S.12) we have

$$\Psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N+1}) \;=\; \sqrt{N+1}\left[\varphi_\gamma(\mathbf{x}_{N+1})\,\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)\right]^{\text{totally symmetrized}}_{\text{with respect to }\mathbf{x}_1,\ldots,\mathbf{x}_{N+1}}. \tag{S.13}$$

In principle, the total symmetrization here amounts to averaging over all $(N+1)!$ permutations of the $(\mathbf{x}_1,\ldots,\mathbf{x}_{N+1})$, but because $\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ is already totally symmetric with respect to the first N particles, it is enough to average over $N+1$ distinct terms only. Thus,

$$\Psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N+1}) \;=\; \frac{\sqrt{N+1}}{N+1}\sum_{i=1} N+1\varphi(\mathbf{x}_i)\times\Psi(\mathbf{x}_1,\ldots,\not{\mathbf{x}}_i,\ldots\mathbf{x}_{N+1}). \tag{5}$$

$\mathcal{Q.E.D.}$

Problem **1**(c):

First, consider a one-body linear operator of the form $\hat{O}=|\alpha\rangle\langle\beta|$. The matrix elements of this operator between generic one-particle wave functions $\Psi_1(\mathbf{x})$ and $\Psi_2(\mathbf{x})$ are given by

$$\langle\Psi_1|\,\hat{O}\,|\Psi_2\rangle \;=\; \langle\Psi_1|\alpha\rangle\langle\beta|\Psi_2\rangle \;=\; \int d^3\mathbf{x}\int d^3\mathbf{x}'\;\Psi_1^*(\mathbf{x})\varphi_\alpha(\mathbf{x})\varphi_\beta^*(\mathbf{x}')\Psi_2(\mathbf{x}'). \tag{S.14}$$

Consequently, in the first-quantized formalism, the matrix elements of the $\hat{O}^{(1)}_{\text{tot}}$ operator between the N–boson quantum states $\langle N,\Psi_1|$ and $|N,\Psi_2\rangle$ are given by

$$\begin{aligned}
&\langle N,\Psi_1|\,\hat{O}^{(1)}_{\text{tot}}\,|N,\Psi_2\rangle \;=\\
&\quad=\sum_{i=1}^{N}\int d^3\mathbf{x}_1\cdots\int d^3\mathbf{x}_N\int d^3\mathbf{x}'_i\;\Psi_1^*(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_N)\varphi_\alpha(\mathbf{x}_i)\varphi_\beta^*(\mathbf{x}'_i)\Psi_2(\mathbf{x}_1,\ldots,\mathbf{x}'_i,\ldots,\mathbf{x}_N)\\
&\quad=N\int d^3\mathbf{x}_1\cdots\int d^3\mathbf{x}_N\int d^3\mathbf{x}'_N\;\Psi_1^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N)\phi_\alpha(\mathbf{x}_N)\phi_\beta^*(\mathbf{x}'_N)\Psi_2(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}'_N)\\
&\quad=\int d^3\mathbf{x}_1\cdots\int d^3\mathbf{x}_{N-1}\;\Psi_1'^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})\;\Psi_2'(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})
\end{aligned} \tag{S.15}$$

where the second equality follows from the total symmetry of the wave functions Ψ_1 and Ψ_2, and on the last line, the $(N-1)$-particle wave functions $\Psi'_{1,2}(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1})$ are defined according to eq. (4) (but using $\hat{a}_\alpha$ instead of $\hat{a}_\gamma$ for the Ψ'_1 and $\hat{a}_\beta$ for the Ψ_2). In light of problem (a), this means

$$\langle N,\Psi_1|\,\hat{O}^{(1)}_{\text{tot}}\,|N,\Psi_2\rangle \;=\; \big\langle (N-1),\Psi'_1|(N-1),\Psi'_2\big\rangle \;=\; \langle N,\Psi_1|\,\hat{a}^\dagger_\alpha\hat{a}_\beta\,|N,\Psi_2\rangle\,. \tag{S.16}$$

On the other hand, in the second quantized formalism $\hat{O}^{(2)}_{\text{tot}}$ is simply $\hat{a}^\dagger_\alpha\hat{a}_\beta$, thus according to eq. (S.16),

$$\langle N,\Psi_1|\,\hat{O}^{(1)}_{\rm tot}\,|N,\Psi_2\rangle \;=\; \langle N,\Psi_1|\,\hat{O}^{(2)}_{\rm tot}\,|N,\Psi_2\rangle \quad \text{for any } \langle N,\Psi_1| \text{ and } |N,\Psi_2\rangle\,. \tag{S.17}$$

Now we need to extend this result to generic one-body operators. Fortunately, any operator $\hat{A}_1$ in the one-particle Hilbert space can be decomposed as $\hat{A}_1 = \sum_{\alpha,\beta} |\alpha\rangle\, A_{\alpha,\beta}\, \langle\beta|$ where $A_{\alpha,\beta}$ are the matrix elements $\langle\alpha|\,\hat{A}_1\,|\beta\rangle$. The definition (6) of first-quantized $\hat{A}^{(1)}_{\rm tot}$ of N particles is obviously linear with respect to the $\hat{A}_1$, thus

$$\hat{A}^{(1)}_{\rm tot} \;=\; \sum_{\alpha,\beta} A_{\alpha,\beta} \sum_{i=1}^{N} \Big(|\alpha\rangle\,\langle\beta|\Big)_{i^{\underline{\rm th}}\text{ particle}} \tag{S.18}$$

and therefore, thanks to eq. (S.17),

$$\langle N,\Psi_1|\,\hat{A}^{(1)}_{\rm tot}\,|N,\Psi_2\rangle \;=\; \sum_{\alpha,\beta} A_{\alpha,\beta}\,\langle N,\Psi_1|\,\hat{a}^\dagger_\alpha \hat{a}_\beta\,|N,\Psi_2\rangle = \; \langle N,\Psi_1|\,\hat{A}^{(2)}_{\rm tot}\,|N,\Psi_2\rangle\,. \qquad \mathcal{Q.E.D.}$$

Problem **1**(d):

Again, we start with a particularly simple 2-body operator $\hat{O}_2 = (|\alpha\rangle \otimes |\beta\rangle)(\langle\gamma| \otimes \langle\delta|)$ which acts on two-particle wave functions according to

$$\langle 2,\Psi_1|\,\hat{O}_2\,|2,\Psi_2\rangle \;=\; \int d^3\mathbf{x}_1 \int d^3\mathbf{x}_2\; \Psi_1^*(\mathbf{x}_1,\mathbf{x}_2)\phi_\alpha(\mathbf{x}_1)\phi_\beta(\mathbf{x}_2) \times \int d^3\mathbf{y}_1 \int d^3\mathbf{y}_2\; \phi_\gamma^*(\mathbf{x}_1)\phi_\delta^*(\mathbf{x}_2)\Psi_2(\mathbf{x}_1,\mathbf{x}_2). \tag{S.19}$$

Consequently, the first-quantized $\hat{O}^{(1)}_{\rm tot}$ operator constructed according to eq. (9) acts in the N-

boson Hilbert space as

$$
\begin{aligned}
\langle N,\Psi_1|\,\hat{O}^{(1)}_{\rm tot}\,|N,\Psi_2\rangle \;=&\\
=\;& \tfrac{1}{2}\sum_{i\neq j}\int d^3\mathbf{x}_1\cdots\int d^3\mathbf{x}_N\;\Psi_1^*(\mathbf{x}_1,\ldots,\mathbf{x}_i,\ldots,\mathbf{x}_j\ldots,\mathbf{x}_N)\phi_\alpha(\mathbf{x}_i)\phi_\beta(\mathbf{x}_j)\times\\
&\qquad\times\int d^3\mathbf{x}'_i\int d^3\mathbf{x}'_j\;\phi^*_\gamma(\mathbf{x}'_i)\phi^*_\delta(\mathbf{x}'_j)\Psi_2(\mathbf{x}_1,\ldots,\mathbf{x}'_i,\ldots,\mathbf{x}'_j,\ldots,\mathbf{x}_N)\\
=\;& \frac{N(N-1)}{2}\int d^3\mathbf{x}_1\cdots\int d^3\mathbf{x}_N\;\Psi_1^*(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2},\mathbf{x}_{N-1},\mathbf{x}_N)\phi_\alpha(\mathbf{x}_{N-1})\phi_\beta(\mathbf{x}_N)\times\\
&\qquad\times\int d^3\mathbf{x}'_{N-1}\int d^3\mathbf{x}'_N\;\phi^*_\gamma(\mathbf{x}_{N-1})\phi^*_\delta(\mathbf{x}_N)\Psi_2(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2},\mathbf{x}'_{N-1},\mathbf{x}'_N)\\
=\;& \tfrac{1}{2}\int d^3\mathbf{x}_1\cdots\int d^3\mathbf{x}_{N-2}\;\Psi_1^{\prime\prime *}(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2})\;\Psi_2''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2})
\end{aligned}
\tag{S.20}
$$

where again the second equality follows from the total symmetry of the bosonic wave functions, and where on the the last line $\Psi_1^{|prime\prime *}$ and Ψ_2'' are $(N-2)$ particle wave functions constructed according to

$$
\Psi_1^{\prime\prime *}(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2}) \;=\; \sqrt{N(N-1)}\int d^3\mathbf{x}_{N-1}\int d^3\mathbf{x}_N\;\Psi_1^*(\mathbf{x}_1,\ldots,\mathbf{x}_N)\varphi_\alpha(\mathbf{x}_{N-1})\varphi_\beta(\mathbf{x}_N), \tag{S.21}
$$

$$
\Psi_2''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-2}) \;=\; \sqrt{N(N-1)}\int d^3\mathbf{x}_{N-1}\int d^3\mathbf{x}_N\;\varphi^*_\gamma(\mathbf{x}_{N-1})\varphi^*_\delta(\mathbf{x}_N)\Psi_2(\mathbf{x}_1,\ldots,\mathbf{x}_N). \tag{S.22}
$$

Notice that the double integral in eq. (S.22) is precisely the integral of eq. (4) applied twice, thus in Fock-space notations

$$
\big|(N-2),\Psi_2''\big\rangle \;=\; \hat{a}_\gamma\hat{a}_\delta\,|N,\Psi_2\rangle\,. \tag{S.23}
$$

As to eq. (S.21), it looks like the complex conjugate of eq. (S.22), hence in Fock-space notations

$$
\big\langle(N-2),\Psi_1''\big| \;=\; \langle N,\Psi_1|\,\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\,. \tag{S.24}
$$

Putting this all together, we arrive at

$$
\langle N,\Psi_1|\,\hat{O}^{(1)}_{\rm tot}\,|N,\Psi_2\rangle \;=\; \tfrac{1}{2}\,\langle N,\Psi_1|\,\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\delta\,|N,\Psi_2\rangle \qquad \text{for } \hat{O}_2=(|\alpha\rangle\otimes|\beta\rangle)(\langle\gamma|\otimes\langle\delta|). \tag{S.25}
$$

To extend this result to a general two-body operator $\hat{B}_2$, we use matrix-element decomposition in the two-distinct-particle Hilbert space:

$$\hat{B}_2 \;=\; \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta} B_{\alpha,\beta,\gamma,\delta}\,\big(|\alpha\rangle\otimes|\beta\rangle\big)\big(\langle\gamma|\otimes\langle\delta|\big) \quad \text{where } B_{\alpha,\beta,\gamma,\delta} \;=\; (\langle\alpha|\otimes\langle\beta|)\hat{B}_2(|\gamma\rangle\otimes|\delta\rangle). \tag{S.26}$$

Consequently, similarly to eq. (S.18), the first-quantized form of $\hat{S}_{\rm tot}$ can be written as

$$\hat{B}^{(1)}_{\rm tot} \;=\; \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta} B_{\alpha,\beta,\gamma,\delta}\sum_{i\neq j}\Big(|\alpha\rangle\,\langle\gamma|\Big)_{i^{\underline{\rm th}}\,\text{particle}} \times \Big(|\beta\rangle\,\langle\delta|\Big)_{j^{\underline{\rm th}}\,\text{particle}}\,, \tag{S.27}$$

and therefore

$$\langle N,\Psi_1|\,\hat{B}^{(1)}_{\rm tot}\,|N,\Psi_2\rangle \;=\; \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta} B_{\alpha,\beta,\gamma,\delta}\times\langle N,\Psi_1|\,\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\delta\,|N,\Psi_2\rangle \;\equiv\; \langle N,\Psi_1|\,\hat{B}^{(2)}_{\rm tot}\,|N,\Psi_2\rangle$$

$\mathcal{Q.E.D.}$

Problem **2**(a):

This is a simple exercise of the Leibniz rule for commutators:

$$\begin{aligned}
[\hat{a}^\dagger_\alpha\hat{a}_\beta,\hat{a}^\dagger_\gamma] \;&=\; [\hat{a}^\dagger_\alpha,\hat{a}^\dagger_\gamma]\hat{a}_\beta \;+\; \hat{a}^\dagger_\alpha[\hat{a}_\beta,\hat{a}^\dagger_\gamma] \;=\; 0 \;+\; \hat{a}^\dagger_\alpha\delta_{\beta,\gamma} \;=\; \delta_{\beta,\gamma}\hat{a}^\dagger_\alpha\,,\\
[\hat{a}^\dagger_\alpha\hat{a}_\beta,\hat{a}_\delta] \;&=\; [\hat{a}^\dagger_\alpha,\hat{a}_\delta]\hat{a}_\beta \;+\; \hat{a}^\dagger_\alpha[\hat{a}_\beta,\hat{a}_\delta] \;=\; -\delta_{\alpha,\delta}\hat{a}_\beta \;+\; 0 \;=\; -\delta_{\alpha,\delta}\hat{a}_\beta\,,\\
[\hat{a}^\dagger_\alpha\hat{a}_\beta,\hat{a}^\dagger_\gamma\hat{a}_\delta] \;&=\; [\hat{a}^\dagger_\alpha\hat{a}_\beta,\hat{a}^\dagger_\gamma]\hat{a}_\delta \;+\; \hat{a}^\dagger_\gamma[\hat{a}^\dagger_\alpha\hat{a}_\beta,\hat{a}_\gamma\hat{a}_\delta] \;=\; \delta_{\beta,\gamma}\hat{a}^\dagger_\alpha\hat{a}_\delta \;-\; \delta_{\alpha,\delta}\hat{a}^\dagger_\gamma\hat{a}_\beta\,.
\end{aligned} \tag{S.28}$$

Problem **2**(b):

Given

$$\hat{A}^{(2)}_{\rm tot} \;=\sum_{\alpha,\beta}\langle\alpha|\,\hat{A}_1\,|\beta\rangle\;\hat{a}^\dagger_\alpha\hat{a}_\beta \tag{S.29}$$

and

$$\hat{B}^{(2)}_{\rm tot} \;=\sum_{\gamma,\delta}\langle\gamma|\,\hat{B}_1\,|\delta\rangle\;\hat{a}^\dagger_\gamma\hat{a}_\delta\,, \tag{S.30}$$

we immediately have

$$\begin{aligned}
\left[\hat{A}^{(2)}_{\text{tot}}, \hat{B}^{(2)}_{\text{tot}}\right] &= \sum_{\alpha,\beta,\gamma,\delta} \langle\alpha|\,\hat{A}_1\,|\beta\rangle\ \langle\gamma|\,\hat{B}_1\,|\delta\rangle\ [\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta] \\
&\langle\!\langle \text{using (S.28)} \rangle\!\rangle \\
&= \sum_{\alpha,\beta,\gamma,\delta} \langle\alpha|\,\hat{A}_1\,|\beta\rangle\ \langle\gamma|\,\hat{B}_1\,|\delta\rangle\ \Big(\delta_{\beta,\gamma}\hat{a}^\dagger_\alpha \hat{a}_\delta\ -\ \delta_{\alpha,\delta}\hat{a}^\dagger_\gamma \hat{a}_\beta\Big) \\
&= \sum_{\alpha,\delta} \hat{a}^\dagger_\alpha \hat{a}_\delta \times \sum_{\beta=\gamma} \langle\alpha|\,\hat{A}_1\,|\gamma\rangle\,\langle\gamma|\,\hat{B}_1\,|\delta\rangle\ - \sum_{\beta,\gamma} \hat{a}^\dagger_\gamma \hat{a}_\beta \times \sum_{\alpha=\delta} \langle\gamma|\,\hat{B}_1\,|\alpha\rangle\,\langle\alpha|\,\hat{A}_1\,|\beta\rangle \\
&= \sum_{\alpha,\delta} \hat{a}^\dagger_\alpha \hat{a}_\delta\ \langle\alpha|\,\hat{A}_1\hat{B}_1\,|\delta\rangle\ - \sum_{\beta,\gamma} \hat{a}^\dagger_\gamma \hat{a}_\beta\ \langle\gamma|\,\hat{B}_1\hat{A}_1\,|\beta\rangle \\
&\langle\!\langle \text{renaming summation indices} \rangle\!\rangle \\
&= \sum_{\alpha,\beta} \hat{a}^\dagger_\alpha \hat{a}_\beta \times \Big(\langle\alpha|\,\hat{A}_1\hat{B}_1\,|\beta\rangle\ -\ \langle\alpha|\,\hat{B}_1\hat{A}_1\,|\beta\rangle\Big) \\
&= \sum_{\alpha,\beta} \hat{a}^\dagger_\alpha \hat{a}_\beta \times \langle\alpha|\,\Big([\hat{A}_1, \hat{B}_1] = \hat{C}_1\Big)\,|\beta\rangle\ \equiv\ \hat{C}^{(2)}_{\text{tot}}.
\end{aligned} \tag{S.31}$$

Problem **2**(c):

Again, we apply the Leibniz rule:

$$\begin{aligned}
[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta] =&\ [\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\alpha]\hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta + \hat{a}^\dagger_\alpha[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}^\dagger_\beta]\hat{a}_\gamma \hat{a}_\delta + \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}_\gamma]\hat{a}_\delta + \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma[\hat{a}^\dagger_\mu \hat{a}_\nu, \hat{a}_\delta] \\
=&\ \delta_{\nu\alpha}\hat{a}^\dagger_\mu \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta + \delta_{\nu\beta}\hat{a}^\dagger_\alpha \hat{a}^\dagger_\mu \hat{a}_\gamma \hat{a}_\delta - \delta_{\mu\gamma}\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\nu \hat{a}_\delta - \delta_{\mu\delta}\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\nu\,.
\end{aligned} \tag{S.32}$$

Problem **2**(d):

In the Fock space,

$$\hat{A}^{(2)}_{\text{tot}}\ =\ \sum_{\mu\, nu} \langle\mu|\,\hat{A}_1\,|\nu\rangle\ \hat{a}^\dagger_\mu \hat{a}_\nu \tag{7}$$

and

$$\hat{B}^{(2)}_{\text{tot}}\ =\ \tfrac{1}{2} \sum_{\alpha,\beta,\gamma,\delta} \langle\alpha\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\delta\rangle\ \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta\,, \tag{10}$$

where $\langle\alpha\otimes\beta|$ is a short-hand for the un-symmetrized two-particle wave function $(\langle\alpha|\otimes\langle\beta|)$ and likewise $|\gamma\otimes\delta\rangle=(|\gamma\rangle\otimes|\delta\rangle)$. Therefore,

$$
\begin{aligned}
[\hat{A}^{(2)}_{\rm tot},\hat{B}^{(2)}_{\rm tot}] \;&=\; \tfrac12\sum_{\mu,\nu,\alpha,\beta,\gamma,\delta}\langle\mu|\,\hat{A}_1\,|\nu\rangle\;\langle\alpha\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\delta\rangle\;[\hat{a}^\dagger_\mu\hat{a}_\nu,\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\delta]\\
&\langle\!\langle\text{using eq. (S.32)}\rangle\!\rangle\\
&=\;\tfrac12\sum_{\mu,\beta,\gamma,\delta}\hat{a}^\dagger_\mu\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\delta\times\sum_\nu\langle\mu|\,\hat{A}_1\,|\nu\rangle\,\langle\nu\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\delta\rangle\\
&\qquad+\;\tfrac12\sum_{\alpha,\mu,\gamma,\delta}\hat{a}^\dagger_\alpha\hat{a}^\dagger_\mu\hat{a}_\gamma\hat{a}_\delta\times\sum_\nu\langle\mu|\,\hat{A}_1\,|\nu\rangle\,\langle\alpha\otimes\nu|\,\hat{B}_2\,|\gamma\otimes\delta\rangle\\
&\qquad-\;\tfrac12\sum_{\alpha,\beta,\nu,\delta}\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\nu\hat{a}_\delta\times\sum_\mu\langle\alpha\otimes\beta|\,\hat{B}_2\,|\mu\otimes\delta\rangle\,\langle\mu|\,\hat{A}_1\,|\nu\rangle\\
&\qquad-\;\tfrac12\sum_{\alpha,\beta,\gamma,\nu}\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\nu\times\sum_\mu\langle\alpha\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\mu\rangle\,\langle\mu|\,\hat{A}_1\,|\nu\rangle\\
&\langle\!\langle\text{renaming summation indices}\rangle\!\rangle\\
&=\;\tfrac12\sum_{\alpha,\beta,\gamma,\delta}\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\delta\times C_{\alpha,\beta,\gamma,\delta}\,,
\end{aligned}
\tag{S.33}
$$

where

$$
\begin{aligned}
C_{\alpha,\beta,\gamma,\delta}\;&=\sum_\lambda\langle\alpha|\,\hat{A}_1\,|\lambda\rangle\,\langle\lambda\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\delta\rangle\;+\;\sum_\lambda\langle\beta|\,\hat{A}_1\,|\lambda\rangle\,\langle\alpha\otimes\lambda|\,\hat{B}_2\,|\gamma\otimes\delta\rangle\\
&\quad-\;\sum_\lambda\langle\alpha\otimes\beta|\,\hat{B}_2\,|\lambda\otimes\delta\rangle\;\langle\lambda|\,\hat{A}_1\,|\gamma\rangle\;-\;\sum_\lambda\langle\alpha\otimes\beta|\,\hat{B}_2\,|\gamma\otimes\lambda\rangle\;\langle\lambda|\,\hat{A}_1\,|\delta\rangle\\
&=\;\langle\alpha\otimes\beta|\,\Big(\hat{A}_1(1^{\underline{\rm st}})\hat{B}_2\;+\;\hat{A}_1(2^{\underline{\rm nd}})\hat{B}_2\;-\;\hat{B}_2\hat{A}_1(1^{\underline{\rm st}})\;-\;\hat{B}_2\hat{A}_1(2^{\underline{\rm nd}})\Big)\,|\gamma\otimes\delta\rangle\\
&=\;\langle\alpha\otimes\beta|\,\Big[\Big(\hat{A}_1(2^{\underline{\rm nd}})+\hat{A}_1(2^{\underline{\rm nd}})\Big)\,,\hat{B}_2\Big]\,|\gamma\otimes\delta\rangle\;\equiv\;\langle\alpha\otimes\beta|\,\hat{C}_2\,|\gamma\otimes\delta\rangle\,.
\end{aligned}
\tag{S.34}
$$

Consequently, $[\hat{A}^{(2)}_{\rm tot},\hat{B}^{(2)}_{\rm tot}]=\hat{C}^{(2)}_{\rm tot}$. $\quad\mathcal{Q.E.D.}$

Problem **3**(a):

Use product-of-exponentials formula

$$
\forall\hat{A},\hat{B}:\quad e^{\hat{A}}e^{\hat{B}}\;=\;\exp\Big(\hat{A}+\hat{B}\;+\;\tfrac12[\hat{A},\hat{B}]\;+\;\tfrac1{12}[(\hat{A}-\hat{B}),[\hat{A},\hat{B}]]\;+\;\cdots\Big)\,.
\tag{S.35}
$$

In particular, for $\hat{A}=\xi\hat{a}^\dagger$, $\hat{B}=\xi^*\hat{a}$ and $[\hat{A},\hat{B}]=\xi\xi^*$ being a c-number,

$$
e^{\xi\hat{a}^\dagger}e^{-\xi^*\hat{a}}\;=\;\exp\Big(\xi\hat{a}^\dagger-\xi^*\hat{a}+\tfrac12\xi\xi^*\Big),\quad\text{exactly,}
\tag{S.36}
$$

and therefore

$$|\xi\rangle \stackrel{\text{def}}{=} e^{\xi\hat{a}^\dagger - \xi^*\hat{a}}\,|0\rangle \;=\; e^{-|\xi|^2/2} e^{\xi\hat{a}^\dagger} e^{-\xi^*\hat{a}}\,|0\rangle \;=\; e^{-|\xi|^2/2} e^{\xi\hat{a}^\dagger}\,|0\rangle\,. \tag{S.37}$$

(Note $e^{-\xi^*\hat{a}}\,|0\rangle = |0\rangle$ since $\hat{a}\,|0\rangle = 0$.)

Next, $[\hat{a},\hat{a}^\dagger] = 1$ implies that for any function $f(\hat{a}^\dagger)$, $[\hat{a}, f(\hat{a}^\dagger)] = f'(\hat{a}^\dagger)$. In particular, $[\hat{a}, e^{\xi\hat{a}^\dagger}] = \xi e^{\xi\hat{a}^\dagger}$ or in other words, $(\hat{a}-\xi)e^{\xi\hat{a}^\dagger} = e^{\xi\hat{a}^\dagger}\hat{a}$ and hence $(\hat{a}-\xi)\,|\xi\rangle \propto e^{\xi\hat{a}^\dagger}\hat{a}\,|0\rangle = 0$. $\mathcal{Q.E.D.}$

Problem **3**(b):

For any *normal-ordered* product of creation and annihilation operators — *i.e.*, a product in which all creation operators are to the right of all annihilation operators — one has $\langle\xi|\,(\hat{a}^\dagger)^k(\hat{a})^\ell\,|\xi\rangle = (\xi^*)^k\xi^\ell$, simply because $\hat{a}\,|\xi\rangle = \xi\,|\xi\rangle$ and $\langle\xi|\,\hat{a}^\dagger = \xi^*\,\langle\xi|$. In particular, $\langle\xi|\,(\hat{n} = \hat{a}^\dagger\hat{a})\,|\xi\rangle = \xi^*\xi$. On the other hand,

$$\hat{n}^2 \;=\; \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} \;=\; \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} \;+\; \hat{a}^\dagger\hat{a} \;\Longrightarrow\; \langle\xi|\,\hat{n}^2\,|\xi\rangle \;=\; (\xi^*)^2\xi^2 \;+\; \xi^*\xi \;=\; \bar{n}^2 \;+\; \bar{n} \tag{S.38}$$

hence $\Delta n = \sqrt{\langle\hat{n}^2\rangle - \bar{n}^2} = \sqrt{\bar{n}}$.

In a similar manner,

$$\hat{q} \;=\; \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger), \quad \hat{q}^2 \;=\; \frac{\hbar}{2m\omega}\Big((\hat{a})^2 + (\hat{a}^\dagger)^2 + 2\hat{a}^\dagger\hat{a} + 1\Big) \;\Longrightarrow$$
$$\langle\xi|\,\hat{q}^2\,|\xi\rangle \;=\; \frac{\hbar}{2m\omega}\big((\xi+\xi^*)^2 + 1\big) \;=\; \langle\xi|\,\hat{q}\,|\xi\rangle^2 \;+\; \frac{\hbar}{2m\omega}$$

and likewise

$$\langle\xi|\,\hat{p}^2\,|\xi\rangle \;=\; \frac{m\omega\hbar}{2}\big((-i\xi + i\xi^*)^2 + 1\big) \;=\; \langle\xi|\,\hat{p}\,|\xi\rangle^2 \;+\; \frac{m\omega\hbar}{2}\,,$$

thus

$$\Delta q \;=\; \sqrt{\frac{\hbar}{2m\omega}}\,, \quad \Delta p \;=\; \sqrt{\frac{m\omega\hbar}{2}}\,, \quad \Delta q\Delta p \;=\; \frac{\hbar}{2}\,. \tag{S.39}$$

$\mathcal{Q.E.D.}$

Problem $\mathbf{3}$(c):

In the Schrödinger picture, $\hat{a}^\dagger$ is time independent, hence $(d/dt)e^{\xi\hat{a}^\dagger} = (d\xi/dt)\hat{a}^\dagger e^{\xi\hat{a}^\dagger}$. Using time independence of the magnitude $|\xi|$, we then have

$$\frac{d}{dt}\left(|\xi\rangle = e^{-|\xi|^2/2}e^{\xi\hat{a}^\dagger}|0\rangle\right) \;=\; \frac{d\xi}{dt}\,\hat{a}^\dagger|\xi\rangle \;=\; \frac{1}{\xi}\frac{d\xi}{dt}\,\hat{a}^\dagger\hat{a}\,|\xi\rangle \;=\; -i\omega\hat{a}^\dagger\hat{a}\,|\xi\rangle \tag{S.40}$$

where the last equality comes from $\xi(t) = \xi_0 e^{-i\omega t}$. In other words,

$$i\hbar\frac{d}{dt}|\xi(t)\rangle \;=\; \hbar\omega\hat{a}^\dagger\hat{a}\,|\xi(t)\rangle \;\equiv\; \hat{H}\,|\xi(t)\rangle\,. \tag{S.41}$$

$\mathcal{Q.E.D.}$

Problem $\mathbf{3}$(d):

In question $\mathbf{3}$(a) we saw that $[\hat{a},\hat{a}^\dagger] = 1$ implies $e^{\xi\hat{a}^\dagger}\hat{a} = (\hat{a}-\xi)e^{\xi\hat{a}^\dagger}$ for any c-number ξ. Iterating this identity gives us $e^{\xi\hat{a}^\dagger}f(\hat{a}) = f(\hat{a}-\xi)e^{\xi\hat{a}^\dagger}$ for any function $f(\hat{a})$ of the annihilation operator, and in particular

$$e^{\xi\hat{a}^\dagger}\,e^{\eta^*\hat{a}} \;=\; e^{\eta^*(\hat{a}-\xi)}\,e^{\xi\hat{a}^\dagger} \;=\; e^{-\eta^*\xi}\,e^{\eta^*\hat{a}}\,e^{\xi\hat{a}^\dagger}. \tag{S.42}$$

Consequently, the quantum overlap of the coherent states $|\xi\rangle$ and $\langle\eta|$ is

$$\begin{aligned}\langle\eta|\xi\rangle \;&=\; e^{-|\eta|^2/2}e^{-|\xi|^2/2}\,\langle 0|\,e^{\eta^*\hat{a}}e^{\xi\hat{a}^\dagger}\,|0\rangle\\ &=\; e^{-|\eta|^2/2}e^{-|\xi|^2/2}e^{-\eta^*\xi}\,\langle 0|\,e^{\xi\hat{a}^\dagger}e^{\eta^*\hat{a}}\,|0\rangle\\ &=\; \exp\left(-\tfrac{1}{2}|\eta|^2 - \tfrac{1}{2}|\xi|^2 - \eta^*\xi\right)\times 1\end{aligned} \tag{S.43}$$

because $e^{\eta^*\hat{a}}|0\rangle = |0\rangle$, $\langle 0|\,e^{\xi\hat{a}^\dagger} = \langle 0|$ and $\langle 0|0\rangle = 1$. In terms of the probability overlap,

$$\left|\langle\eta|\xi\rangle\right|^2 \;=\; e^{-|\eta-\xi|^2}. \tag{S.44}$$

Problem **3**(e):

Generalization of coherent states to multi-oscillatory systems and further to the creation / annihilation fields is completely straightforward:

$$|\text{coherent}\rangle \stackrel{\text{def}}{=} \exp\bigl(\hat{F}^\dagger - \hat{F}\bigr)|0\rangle = e^{-\bar{N}/2}\,e^{\hat{F}^\dagger}|0\rangle \tag{S.45}$$

where

$$\hat{F}^\dagger = \xi\hat{a}^\dagger \to \sum_\alpha \xi_\alpha \hat{a}^\dagger_\alpha \to \int d^3\mathbf{x}\,\Phi(\mathbf{x})\hat{\Psi}^\dagger(\mathbf{x}). \tag{S.46}$$

Similar to the single-oscillator theory, $\bigl(\hat{\Psi}(\mathbf{x})-\Phi(\mathbf{x})\bigr)e^{\hat{F}^\dagger} = e^{\hat{F}^\dagger}\hat{\Psi}^\dagger(\mathbf{x})$, hence

$$\hat{\Psi}(\mathbf{x})|\Phi\rangle = \Phi(\mathbf{x})|\Phi\rangle\,. \tag{S.47}$$

Problem **3**(f):

Using eq. (S.47) and its hermitian conjugate, we have

$$\langle\Phi|\,\hat{\Psi}^\dagger(\mathbf{x}_1)\cdots\hat{\Psi}^\dagger(\mathbf{x}_k)\hat{\Psi}(\mathbf{y}_1)\cdots\hat{\Psi}(\mathbf{y}_\ell)\,|\Phi\rangle = \Phi^*(\mathbf{x}_1)\cdots\Phi^*(\mathbf{x}_k)\Phi(\mathbf{y}_1)\cdots\Phi(\mathbf{y}_\ell) \tag{S.48}$$

for any *normal-ordered* product of the quantum fields. Specifically, for the particle-number operator $\hat{N}$ we have eq. (12), while for its square — whose normal-ordered form

$$\hat{N}^2 = \iint d^3\mathbf{x}\,d^3\mathbf{y}\,\hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}^\dagger(\mathbf{y})\hat{\Psi}(\mathbf{x})\hat{\Psi}(\mathbf{y}) + \int d^3\mathbf{x}\,\hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}(\mathbf{x}) \tag{S.49}$$

generalizes eq. (S.38) — we have

$$\langle\Phi|\,\hat{N}^2\,|\Phi\rangle = \iint d^3\mathbf{x}\,d^3\mathbf{y}\,\Phi^*(\mathbf{x})\Phi^*(\mathbf{y})\Phi(\mathbf{x})\Phi(\mathbf{y}) + \int d^3\mathbf{x}\,\Phi^*(\mathbf{x})\Phi(\mathbf{x}) = \langle\Phi|\,\hat{N}\,|\Phi\rangle^2 + \langle\Phi|\,\hat{N}\,|\Phi\rangle\,, \tag{S.50}$$

and hence $\Delta N = \sqrt{\bar{N}}$, $\mathcal{Q.E.D.}$

Problem **3**(g):

First of all, if $\Phi(\mathbf{x},t)$ satisfies the classical field equation — which looks exactly like a one-particle Schrödinger equation — then $\bar{N}$ remains constant. (This is undergraduate-level QM.) Also, in the Schrödinger picture of the QFT,

$$\frac{d}{dt}\,e^{\hat{F}^\dagger} \;=\; \frac{d\hat{F}^\dagger}{dt}\,e^{\hat{F}^\dagger} \;=\; \left[\int\!d^3\mathbf{x}\,\frac{\partial\Phi(\mathbf{x},t)}{\partial t}\,\hat{\Psi}^\dagger(\mathbf{x})\right]e^{\hat{F}^\dagger} \tag{S.51}$$

thanks to mutual commutativity of the creation fields. Consequently, exactly as in question (c),

$$\begin{aligned}
i\hbar\frac{d}{dt}\left(|\Phi\rangle = e^{-\bar{N}/2}e^{\hat{F}^\dagger}\,|0\rangle\right) \;&=\; \left[\int\!d^3\mathbf{x}\,i\hbar\frac{\partial\Phi(\mathbf{x},t)}{\partial t}\,\hat{\Psi}^\dagger(\mathbf{x})\right]|\Phi\rangle \\
&\langle\!\langle\text{using the classical field equation for }\Phi\rangle\!\rangle \\
&= \left[\int\!d^3\mathbf{x}\left(\frac{-\hbar^2}{2M}\nabla^2+V(\mathbf{x})\right)\Phi(\mathbf{x})\,\hat{\Psi}^\dagger(\mathbf{x})\right]|\Phi\rangle \\
&\langle\!\langle\text{using eq. (14)}\rangle\!\rangle \\
&= \left[\int\!d^3\mathbf{x}\,\hat{\Psi}^\dagger(\mathbf{x})\left(\frac{-\hbar^2}{2M}\nabla^2+V(\mathbf{x})\right)\hat{\Psi}(\mathbf{x})\right]|\Phi\rangle \\
&= \hat{H}\,|\Phi\rangle\,.
\end{aligned} \tag{S.52}$$

$\mathcal{Q.E.D.}$

Problem **3**(h):

Generalizing (d) to multi-oscillatory systems is completely straightforward:

$$|\langle\eta|\xi\rangle|^2 \;=\; \prod_\alpha e^{-|\xi_\alpha-\eta_\alpha|^2} \;=\; \exp\left(-\sum_\alpha|\xi_\alpha-\eta_\alpha|^2\right)$$

or for the field theory,

$$|\langle\Phi_1|\Phi_2\rangle|^2 \;=\; \exp\left(\int\!d^3\mathbf{x}\;|\Phi_1(\mathbf{x})-\Phi_2(\mathbf{x})|^2\right), \tag{S.53}$$

which is exponentially small for any macroscopic $\delta\Phi(\mathbf{x}) = \Phi_1(x)-\Phi_2(x)$. Indeed, a *macroscopic* difference between two coherent states means (by definition) that $\delta\Phi$ affects a large number of particles, $\int|\delta\Phi|^2 \gg 1$ and hence an *exponentially* tiny overlap (S.53).