Problem **1**(a):

The linear sigma model has scalar potential

$$V(\sigma,\underset{\sim}{\pi}) \;=\; \frac{\lambda}{8}\left(\sigma^2+\underset{\sim}{\pi}^2-f^2\right)^2 \;-\; \beta\sigma. \tag{S.1}$$

Any local minimum of this potential satisfies

$$\begin{aligned} \frac{\partial V}{\partial\underset{\sim}{\pi}} \;&=\; \frac{\lambda}{2}\left(\sigma^2+\underset{\sim}{\pi}^2-f^2\right)\times\underset{\sim}{\pi} \;=\; 0\\ \text{and}\quad \frac{\partial V}{\partial\sigma} \;&=\; \frac{\lambda}{2}\left(\sigma^2+\underset{\sim}{\pi}^2-f^2\right)\times\sigma \;-\; \beta \;=\; 0, \end{aligned} \tag{S.2}$$

which together imply $\underset{\sim}{\pi}=0$ while σ satisfies a cubic equation

$$\sigma(\sigma^2-f^2) \;-\; \frac{2\beta}{\lambda} \;=\; 0. \tag{S.3}$$

For small $\beta>0$ this equation has three real solutions, namely

$$\sigma_1 \;\approx\; -\frac{2\beta}{\lambda f^2}\,, \qquad \sigma_2 \;\approx\; -f \;+\; \frac{\beta}{\lambda f^2}\,, \qquad \sigma_3 \;\approx\; +f \;+\; \frac{\beta}{\lambda f^2}\,. \tag{S.4}$$

Furthermore, at $\underset{\sim}{\pi}=0$ the second derivatives of the potential are

$$\frac{\partial^2 V}{\partial\sigma^2} \;=\; \lambda(3\sigma^2-f^2), \qquad \frac{\partial^2 V}{\partial\pi_i\partial\pi_j} \;=\; \lambda(\sigma^2-f^2)\delta_{ij}\,, \qquad \frac{\partial^2 v}{\partial\sigma\partial\pi_i} \;=\; 0, \tag{S.5}$$

which means that the stationary point σ_1 is the local maximum of $V(\sigma,\underset{\sim}{\pi})$, the stationary point σ_2 is a saddle point, and only the σ_3 is a minimum. In other words, $V(\sigma,\underset{\sim}{\pi})$ has a unique minimum at $\underset{\sim}{\pi}=0, \sigma=\sigma_3$, exactly as in eq. (2). $\mathcal{Q.E.D.}$

Problem $\mathbf{1}$(b):

Let us shift the σ field by its vacuum expectation value $\sigma(x) = \langle\sigma\rangle + \tilde{\sigma}(x)$. Expanding the scalar potential into powers of the $\underset{\sim}{\pi}$ and $\tilde{\sigma}$ fields, we have

$$\begin{aligned} V(\underset{\sim}{\pi}, \tilde{\sigma}) &= \frac{\lambda}{8}\Big(\underset{\sim}{\pi}^2 + (\langle\sigma\rangle + \tilde{\sigma})^2 - f^2\Big)^2 - \beta(\langle\sigma\rangle + \tilde{\sigma}) \\ &= \text{const} + V_2 + V_3 + V_4 \end{aligned} \tag{S.6}$$

where

$$\begin{aligned} V_4 &= \frac{\lambda}{8} \times \left(\underset{\sim}{\pi}^2 + \tilde{\sigma}^2\right)^2, \\ V_3 &= \frac{\lambda\langle\sigma\rangle}{2} \times \tilde{\sigma} \times (\underset{\sim}{\pi}^2 + \tilde{\sigma}^2) \\ V_2 &= \frac{\lambda}{4}(\langle\sigma\rangle^2 - f^2) \times \underset{\sim}{\pi}^2 + \frac{\lambda}{4}(3\langle\sigma\rangle^2 - f^2) \times \tilde{\sigma}^2 \\ &= \frac{\beta}{2\langle\sigma\rangle} \times \underset{\sim}{\pi}^2 + \left(\frac{\beta}{2\langle\sigma\rangle} + \frac{\lambda\langle\sigma\rangle^2}{2}\right) \times \tilde{\sigma}^2, \end{aligned} \tag{S.7}$$

and the last equality follows from eq. (S.2) for the $\langle\sigma\rangle$. Consequently, we expand the while Lagrangian into powers of $\underset{\sim}{\pi}(x)$ and $\tilde{\sigma}(x)$ and write

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} + \text{const} \tag{S.8}$$

where

$$\begin{aligned} \mathcal{L}_{\text{free}} &= \mathcal{L}_{\text{kin}} - V_2(\underset{\sim}{\pi}, \tilde{\sigma}) \\ &= \left[\tfrac{1}{2}(\partial_\mu\tilde{\sigma})^2 - \left(\frac{\beta}{2\langle\sigma\rangle} + \frac{\lambda\langle\sigma\rangle^2}{2}\right) \times \tilde{\sigma}^2\right] + \left[\tfrac{1}{2}(\partial_\mu\underset{\sim}{\pi})^2 - \frac{\beta}{2\langle\sigma\rangle} \times \underset{\sim}{\pi}^2\right] \end{aligned} \tag{S.9}$$

describes the free $\underset{\sim}{\pi}$ and $\tilde{\sigma}$ fields, and

$$L_{\text{int}} = -V_3(\underset{\sim}{\pi}, \tilde{\sigma}) - V_4(\underset{\sim}{\pi}, \tilde{\sigma}) \tag{S.10}$$

describes their interactions. The particles' masses follow from the quadratic Lagrangian (S.9):

$$\begin{aligned} M_\pi^2 &= \frac{\beta}{\langle\sigma\rangle} \approx \frac{\beta}{f}, \\ \text{and}\quad M_\sigma^2 &= M_\pi^2 + \lambda\langle\sigma\rangle^2 \approx \lambda f^2 \gg M_\pi^2 . \end{aligned} \qquad \mathcal{Q.E.D.}$$

Problem $\mathbf{2}$(a):

Combining definitions (4) with commutation relations (3), we immediately calculate

$$[\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}] \;=\; \cosh(t_{\mathbf{k}}) \sinh(t_{\mathbf{k}'}) \delta_{\mathbf{k},-\mathbf{k}'} \;-\; \sinh(t_{\mathbf{k}}) \cosh(t_{\mathbf{k}'}) \delta_{-\mathbf{k},\mathbf{k}'} \;=\; 0 \qquad \text{(S.11)}$$

where the second equality follows from $t_{\mathbf{k}} = t_{\mathbf{k}'}$ for $\mathbf{k} = -\mathbf{k}'$. Likewise, $[\hat{b}^\dagger_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}] = 0$. Finally,

$$\begin{aligned} [\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}] \;&=\; \cosh(t_{\mathbf{k}}) \cosh(t_{\mathbf{k}'}) \delta_{\mathbf{k},\mathbf{k}'} \;-\; \sinh(t_{-\mathbf{k}}) \sinh(t_{-\mathbf{k}'}) \delta_{-\mathbf{k},-\mathbf{k}'} \\ &=\; \delta_{\mathbf{k},\mathbf{k}'} \Big(\cosh^2(t_{\mathbf{k}}) - \sinh^2(t_{\mathbf{k}}) = 1 \Big). \end{aligned} \qquad \text{(S.12)}$$

In other words, the $\hat{b}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$ operators satisfy the same bosonic commutations relations

$$[\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}] \;=\; 0, \quad [\hat{b}^\dagger_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}] \;=\; 0, \quad [\hat{b}_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}'}] \;=\; \delta_{\mathbf{k},\mathbf{k}'} \,. \qquad \text{(S.13)}$$

as the original $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ operators. $\mathcal{Q.E.D.}$

Problem $\mathbf{2}$(b):

A straightforward calculation shows that

$$\sum_{\mathbf{k}} \omega_{\mathbf{k}} \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} \;= \sum_{\mathbf{k}} \omega_{\mathbf{k}} \cosh(2t_{\mathbf{k}}) \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} \;+\; \tfrac{1}{2} \sum_{\mathbf{k}} \omega_{\mathbf{k}} \sinh(2t_{\mathbf{k}}) \Big(\hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{-\mathbf{k}} \;+\; \hat{a}_{-\mathbf{k}} \hat{a}_{\mathbf{k}} \Big) \;+\; \text{const.} \quad \text{(S.14)}$$

Therefore, the Hamiltonian (5) can be "diagonalized" in terms of the transformed creation / annihilation operators (4) if and only if we can solve for $\omega_{\mathbf{k}}$ and $t_{\mathbf{k}}$ such that

$$\omega_{\mathbf{k}} \cosh(2t_{\mathbf{k}}) \;=\; A_{\mathbf{k}} \quad \text{and} \quad \omega_{\mathbf{k}} \sinh(2t_{\mathbf{k}}) \;=\; B_{\mathbf{k}} \,. \qquad \text{(S.15)}$$

The latter equations are solvable whenever $A_{\mathbf{k}} > |B_{\mathbf{k}}|$ and the solution is

$$t_{\mathbf{k}} \;=\; \frac{1}{2} \operatorname{artanh} \frac{B_{\mathbf{k}}}{A_{\mathbf{k}}} \quad \text{and} \quad \omega_{\mathbf{k}} \;=\; \sqrt{A^2_{\mathbf{k}} - B^2_{\mathbf{k}}} \,. \qquad \text{(S.16)}$$

$\mathcal{Q.E.D.}$

Problem $\mathbf{2}$(c):

In terms of the momentum modes $\tilde{a}_{\mathbf{k}}$ and $\tilde{a}^\dagger_{\mathbf{k}}$ of the shifted fields $\hat{\varphi}(\mathbf{x})$ and $\hat{\varphi}^\dagger(\mathbf{x})$, the free Hamiltonian (8) becomes

$$\hat{H}_{\rm free} \;=\; \sum_{\mathbf{k}} \left[\left(\frac{\mathbf{k}^2}{2M} \,+\, \lambda n \right) \tilde{a}^\dagger_{\mathbf{k}} \tilde{a}_{\mathbf{k}} \;+\; \frac{\lambda n}{2} \left(\tilde{a}_{\mathbf{k}} \tilde{a}_{-\mathbf{k}} \,+\, \tilde{a}^\dagger_{-\mathbf{k}} \tilde{a}^\dagger \mathbf{k} \right) \right]. \tag{S.17}$$

This Hamiltonian has form (5) in term of the shifted annihilation and creation operators $\tilde{a}_{\mathbf{k}} = \hat{a}_{\mathbf{k}} + \sqrt{N}\delta_{bp,\mathbf{0}}$ and $\tilde{a}^\dagger_{\mathbf{k}} = \hat{a}^\dagger_{\mathbf{k}} + \sqrt{N}\delta_{bp,\mathbf{0}}$, which obviously satisfy the same bosonic commutation relations as the un-shifted operators $\hat{a}_{\mathbf{k}}$ and $\tilde{a}_{\mathbf{k}}$. Therefore, we may diagonalize this Hamiltonian by means of the Bogolyubov transform

$$\begin{aligned} \hat{b}_{\mathbf{k}} \;&=\; \cosh(t_{\mathbf{k}}) \times \tilde{a}_{\mathbf{k}} \,+\, \sinh(t_{\mathbf{k}}) \times \tilde{a}^\dagger_{-\mathbf{k}}\,, \\ \hat{b}^\dagger_{\mathbf{k}} \;&=\; \cosh(t_{\mathbf{k}}) \times \tilde{a}^\dagger_{\mathbf{k}} \,+\, \sinh(t_{\mathbf{k}}) \times \tilde{a}_{-\mathbf{k}}\,, \end{aligned} \tag{$\tilde{4}$}$$

for suitable parameters $t_{\mathbf{k}} = t_{-\mathbf{k}}$. Specifically, according to eqs. (S.16), we let

$$t_{\mathbf{k}} \;=\; \frac{1}{2}\,\mathrm{artanh}\,\frac{\lambda n}{\frac{\mathbf{k}^2}{2M} \,+\, \lambda n} \;=\; \frac{1}{4}\log\left(1 \,+\, \frac{4\lambda n M}{\mathbf{k}^2}\right), \tag{S.18}$$

and then the free Hamiltonian (S.17) takes form (9) for

$$\omega_{\mathbf{k}} \;=\; \sqrt{\left(\frac{\mathbf{k}^2}{2M} \,+\, \lambda n\right)^2 \,-\, \Bigl(\lambda n\Bigr)^2} \;=\; |\mathbf{k}| \times \sqrt{\frac{\lambda n}{M} \,+\, \frac{\mathbf{k}^2}{4M^2}}\,. \tag{10}$$

$\mathcal{Q.E.D.}$

Problem $\mathbf{2}(*)$:

The key to this exercise is the multiple commutator formula: For any two operators $\hat{F}$ and $\hat{G}$,

$$\begin{aligned} e^{+\hat{F}}\hat{G}e^{-\hat{F}} \;&=\; \sum_{n=0}^{\infty} \frac{1}{n!}\,[\hat{F},\cdots[\hat{F},\hat{G}]\cdots]_{n\ \rm times} \\ &=\; \hat{G} \,+\, [\hat{F},\hat{G}] \,+\, \tfrac{1}{2}[\hat{F},[\hat{F},\hat{G}]] \,+\tfrac{1}{6}[\hat{F},[\hat{F},[\hat{F},\hat{G}]]] \,+\cdots. \end{aligned} \tag{S.19}$$

Now, let $\hat{F}$ be as in eq. (11) and let $\hat{G}$ be one of the $\tilde{a}_{\mathbf{k}}$ or $\tilde{a}^\dagger_b p$. The simple commutators $[\hat{F},\hat{G}]$

follow directly from the bosonic commutation relations:

$$[\hat{F}, \tilde{a}_{\mathbf{k}}] \;=\; t_{\mathbf{k}} \tilde{a}^{\dagger}_{-\mathbf{k}} \qquad \text{and} \quad [\hat{F}, \tilde{a}^{\dagger}_{\mathbf{k}}] \;=\; t_{\mathbf{k}} \tilde{a}_{-\mathbf{k}} \,. \tag{S.20}$$

Consequently,

$$[\hat{F}, \cdots [\hat{F}, \tilde{a}_{\mathbf{k}}] \cdots]_{n \text{ times}} \;=\; (t_{\mathbf{k}})^n \times \begin{cases} \tilde{a}_{\mathbf{k}} & \text{for even } n, \\ \tilde{a}^{\dagger}_{-\mathbf{k}} & \text{for odd } n, \end{cases} \tag{S.21}$$

and therefore eq. (S.19) yields

$$\begin{aligned} e^{+\hat{F}} \tilde{a}_{\mathbf{k}} e^{-\hat{F}} \;&=\; \sum_{\text{even } n} \frac{(t_{\mathbf{k}})^n}{n!} \times \tilde{a}_{\mathbf{k}} \;+\; \sum_{\text{odd } n} \frac{(t_{\mathbf{k}})^n}{n!} \times \tilde{a}^{\dagger}_{-\mathbf{k}} \\ &=\; \cosh(t_{\mathbf{k}}) \times \tilde{a}_{\mathbf{k}} \;+\; \sinh(t_{\mathbf{k}}) \times \tilde{a}^{\dagger}_{-\mathbf{k}} \\ &=\; \hat{b}_{\mathbf{k}} \,. \end{aligned} \tag{S.22}$$

And since $\hat{F}$ is anti-Hermitian, we also have

$$e^{+\hat{F}} \tilde{a}^{\dagger}_{\mathbf{k}} e^{-\hat{F}} \;=\; \left(e^{+\hat{F}} \tilde{a}_{\mathbf{k}} e^{-\hat{F}} \right)^{\dagger} \;=\; \hat{b}^{\dagger}_{\mathbf{k}} \,, \tag{S.23}$$

and hence the bosonic commutation relations (S.13) follow from the relations (3) via unitary equivalence (S.22) and (S.23).

Finally, for the quantum state $|\Omega_2\rangle = e^{+\hat{F}} |\text{coh}\rangle$ we have

$$\hat{b}_{\mathbf{k}} |\Omega_2\rangle \;=\; \left(e^{+\hat{F}} \tilde{a}_{\mathbf{k}} e^{-\hat{F}} \right) \left(e^{+\hat{F}} |\text{coh}\rangle \right) \;=\; e^{+\hat{F}} \times \tilde{a}_{\mathbf{k}} |\text{coh}\rangle \;=\; 0$$

because $\tilde{a}_{\mathbf{k}} |\text{coh}\rangle \,=\, 0$ for all momenta $\mathbf{k}$. Therefore, from the point of view of quasi-particles created by the $\hat{b}^{\dagger}_{\mathbf{k}}$ operators and annihilated by the $\hat{b}_{\mathbf{k}}$, the $|\Omega_2\rangle$ is the *vacuum state*.

Problem $\mathbf{2}$(d):

In terms of atomic creation and annihilation operators

$$\hat{\mathbf{P}}_{\text{tot}} \;=\; \sum_{\mathbf{k}} \mathbf{k} \times \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} = \sum_{\mathbf{k}\neq\mathbf{0}} \mathbf{k} \times \tilde{a}^\dagger_{\mathbf{k}} \tilde{a}_{\mathbf{k}}\,, \tag{S.24}$$

hence

$$[\hat{\mathbf{P}}_{\text{tot}}, \tilde{a}^\dagger_{\mathbf{k}}] \;=\; +\mathbf{k} \times \tilde{a}^\dagger_{\mathbf{k}} \qquad \text{and} \qquad [\hat{\mathbf{P}}_{\text{tot}}, \tilde{a}_{\mathbf{k}}] \;=\; -\mathbf{k} \times \tilde{a}_{\mathbf{k}}\,. \tag{S.25}$$

Consequently, for any $t_{\mathbf{k}}$ coefficients, the Bogolyubov-transformed operators ($\tilde{4}$) operators satisfy

$$[\hat{\mathbf{P}}_{\text{tot}}, \hat{b}^\dagger_{\mathbf{k}}] \;=\; +\mathbf{k} \times \hat{b}^\dagger_{\mathbf{k}} \qquad \text{and} \qquad [\hat{\mathbf{P}}_{\text{tot}}, \hat{b}_{\mathbf{k}}] \;=\; -\mathbf{k} \times \hat{b}_{\mathbf{k}}\,, \tag{S.26}$$

which means that the quasiparticle created by the $\hat{b}^\dagger_{\mathbf{k}}$ operator and annihilated by the $\hat{b}_{\mathbf{k}}$ has definite momentum $\mathbf{k}$.

Also, straightforward algebra shows that for any $\mathbf{k} \neq \mathbf{0}$ and $t_{\mathbf{k}} = t_{-\mathbf{k}}$

$$\hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} \;-\; \hat{b}^\dagger_{-\mathbf{k}} \hat{b}_{-\mathbf{k}} \;=\; \tilde{a}^\dagger_{\mathbf{k}} \tilde{a}_{\mathbf{k}} \;-\; \tilde{a}^\dagger_{-\mathbf{k}} \tilde{a}_{-\mathbf{k}}\,. \tag{S.27}$$

Consequently,

$$\begin{aligned} \hat{\mathbf{P}}_{\text{tot}} \;&=\; \sum_{\mathbf{k}\neq\mathbf{0}} \mathbf{k} \times \tilde{a}^\dagger_{\mathbf{k}} \tilde{a}_{\mathbf{k}} \\ &=\; \sum_{\mathbf{k}\neq\mathbf{0}} \frac{\mathbf{k}}{2} \Big(\tilde{a}^\dagger_{\mathbf{k}} \tilde{a}_{\mathbf{k}} \;-\; \tilde{a}^\dagger_{-\mathbf{k}} \tilde{a}_{-\mathbf{k}} \Big) \\ &=\; \sum_{\mathbf{k}\neq\mathbf{0}} \frac{\mathbf{k}}{2} \Big(\hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} \;-\; \hat{b}^\dagger_{-\mathbf{k}} \hat{b}_{-\mathbf{k}} \Big) \\ &=\; \sum_{\mathbf{k}\neq\mathbf{0}} \mathbf{k} \times \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}}\,. \end{aligned} \tag{S.28}$$

In other words, the total mechanical momentum of the superfluid helium is simply the net momentum of all the quasi-particles.

Problem **2**(e):

Combining eqs. (S.18) and ($\tilde{4}$) we immediately see that for large momenta **k**,

$$t_{\mathbf{k}} \approx \frac{\lambda n M}{\mathbf{k}^2} \ll 1 \implies \cosh(t_{\mathbf{k}}) \approx 1, \quad \sinh(t_{\mathbf{k}}) \approx t_{\mathbf{k}} \ll 1, \tag{S.29}$$

and therefore

$$\hat{b}^\dagger_{\mathbf{k}} \approx \tilde{a}^\dagger_{\mathbf{k}} + t_{\mathbf{k}}\tilde{a}_{-\mathbf{k}} \approx \tilde{a}^\dagger_{\mathbf{k}} = \hat{a}^\dagger_{\mathbf{k}}. \tag{S.30}$$

Thus, quasi-particles with large momenta are approximately atoms.

On the other hand, for small momenta **k** (but $\mathbf{k} \neq \mathbf{0}$),

$$t_{\mathbf{k}} \approx \frac{1}{4}\log\frac{4\lambda n M}{\mathbf{k}^2} \gg 1 \implies \cosh(t_{\mathbf{k}}), \approx \sinh(t_{\mathbf{k}}) \approx \tfrac{1}{2}e^{t_{\mathbf{k}}} \approx \tfrac{1}{2}\sqrt[4]{\frac{4\lambda n M}{\mathbf{k}^2}} \gg 1, \tag{S.31}$$

and therefore

$$\hat{b}^\dagger_{\mathbf{k}} \approx \tfrac{1}{2}\sqrt[4]{\frac{4\lambda n M}{\mathbf{k}^2}} \times \left(\tilde{a}^\dagger_{\mathbf{k}} + \tilde{a}_{-\mathbf{k}}\right). \tag{S.32}$$

At the same time, the density fluctuations in the superfluid are measured by the operators

$$\widehat{\delta n}(\mathbf{x}) \equiv \hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}(\mathbf{x}) - \langle n\rangle = \sqrt{\langle n\rangle}\left(\hat{\varphi}^\dagger(\mathbf{x}) + \hat{\varphi}(\mathbf{x})\right) + \hat{\varphi}^\dagger(\mathbf{x})\hat{\varphi}(\mathbf{x}). \tag{S.33}$$

Approximating this formula by its leading first-order terms and then taking a Fourier transform, we find

$$\widehat{\delta n}_{\mathbf{k}} \approx \sqrt{n}\left(\tilde{a}^\dagger_{\mathbf{k}} + \tilde{a}_{-\mathbf{k}}\right). \tag{S.34}$$

Hence, the physical meaning of eq. (S.32) is

$$\text{for small } \mathbf{k}, \quad \hat{b}^\dagger_{\mathbf{k}} \approx (\text{coeff}) \times \widehat{\delta n}_{\mathbf{k}}, \tag{S.35}$$

or in other words, for small momenta, the quasiparticle created by the $\hat{b}^\dagger_{\mathbf{k}}$ operator is the density wave in the superfluid, *i.e.* a phonon.

Problem **3**(f):

Non-relativistic mechanics — classical or quantum — is invariant under Galilean boosts, which act on coordinates, momenta and energy according to

$$t' = t, \quad \mathbf{x}'_i = \mathbf{x}_i + \mathbf{v}t, \quad \mathbf{p}'_i = \mathbf{p}_i + m_i\mathbf{v}, \quad H' = H + \mathbf{v}\cdot\mathbf{P}_{\text{total}} + \text{const.} \tag{S.36}$$

A superfluid flowing at a uniform velocity $\mathbf{v}$ is related to the same superfluid at rest via a Galilean boost. Consequently, the lab-frame Hamiltonian of the flowing superfluid is

$$\begin{aligned} \big(\hat{H} - \mu\hat{N}\big)^{\text{lab}}_{\text{frame}} &= \big(\hat{H} - \mu\hat{N}\big)^{\text{fluid}}_{\text{frame}} + \mathbf{v}\cdot\hat{\mathbf{P}}^{\text{fluid}}_{\text{frame}} + \text{const} \\ &= \big(\hat{H}_{\text{free}} + \mathbf{v}\cdot\hat{\mathbf{P}}\big)^{\text{fluid}}_{\text{frame}} + \hat{H}_{\text{int}} + \text{const} \end{aligned} \tag{S.37}$$

where

$$\hat{H}^{\text{lab frame}}_{\text{free}} = \big(\hat{H}_{\text{free}} + \mathbf{v}\cdot\hat{\mathbf{P}}\big)^{\text{fluid}}_{\text{frame}} = \sum_{\mathbf{k}} (\omega_{\mathbf{k}} + \mathbf{v}\cdot\mathbf{k})\, \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\,. \tag{S.38}$$

(The quasiparticle momenta $\mathbf{k}$ in this formula are in the fluid's rest frame.)

Eq. (S.38) applies to a flow of an ideal gas just as well as a flow of a superfluid, the only difference being in the dispersion relations $\omega(\mathbf{k})$. For the ideal gas, $\omega = \mathbf{k}^2/2M$, and therefore for any non-zero velocity $\mathbf{v}$, there are modes $\mathbf{k}$ for which

$$\omega_{\mathbf{k}} + \mathbf{v}\cdot\mathbf{k} = \frac{(\mathbf{k} - M\mathbf{v})^2}{2M} - \tfrac{1}{2}M\mathbf{v}^2 < 0. \tag{S.39}$$

For such modes, excitations created by the $\hat{b}^\dagger_{\mathbf{k}}$ have negative energy, which means that any perturbation of the flowing gas can create such negative-energy excitations and slow down the flow.

For the superfluid however, $\omega_{\mathbf{k}} \geq v_c|\mathbf{k}|$ for all modes, hence

$$\omega_{\mathbf{k}} + \mathbf{v}\cdot\mathbf{k} \geq (v_c - |\mathbf{v}|)|\mathbf{k}| \geq 0 \tag{S.40}$$

for all the excitation modes, provided $|\mathbf{v}| \leq v_0$. Consequently, *all* excitations in the flowing superfluid have positive lab-frame energies, so they don't get spontaneously created and the flow persists forever without dissipation, thus *superfluidity.*